POMS PRODUCTION AND OPERATIONS MANAGEMENT

PRODUCTION AND OPERATIONS MANAGEMENT POMS

Vol. 24, No. 7, July 2015, pp. 1033–1053
ISSN 1059-1478 | EISSN 1937-5956 | 15 | 2407 | 1033

DOI 10.1111/poms.12370

Inventory Planning for a Modular Product Family

Anand Paul
Department of Information Systems & Operations Management, Warrington College of Business Administration, University of Florida, Gainesville, Florida 32611-7169, USA, paulaa@ufl.edu

Yinliang (Ricky) Tan
A. B. Freeman School of Business, Tulane University, New Orleans, Louisiana 70118, USA, tanyinliang@gmail.com

Asoo J. Vakharia
Department of Information Systems & Operations Management, Warrington College of Business Administration, University of Florida, Gainesville, Florida 32611-7169, USA, asoov@ufl.edu

This paper is motivated by observing that an increasing number of firms are offering modular products assembled with multiple option choices for the consumer. Starting with the PC offerings by Dell which allowed (and still allows) users to configure their product by choosing among multiple choices for each option, the current market place seems to have evolved to a make-to-stock scenario where Apple offers its IPAD series with multiple models each with a unique storage size, color, and wireless chip technology. The focus of our work is on determining the optimal stocking level of modular end-products. Our analysis is based on a benchmark model with the aim of maximizing expected profit subject to an aggregate fill rate constraint as well as variant-specific individual fill rates under a make-to-stock setting. To further assess the robustness of our finding, we consider the extensions of correlated market preferences over options, price-dependent demand, and alternative probability distributions for characterizing uncertainty in market preferences or aggregate demand. Finally we also show how to extend the single period model into a multiple-period setting. Through extensive computational analysis, we find that more precise estimates of market preferences for various modular options constitute extremely valuable information that goes beyond the usefulness of forecasts of aggregate market demand. From a practical perspective, this might be indicative of another classic marketing-operations trade-off. Offering more options for consumers would be preferred by marketing managers since this would reach more consumers and hence, enhance product sales. On the other hand, the ability to obtaining greater forecast accuracy would decline when the number of options increase. Hence, from an operational perspective, it would be preferred to limit option choices (so that better forecasts can be obtained) since this would lead to lower stocking costs and hence, higher profits.

Key words: modular products; inventory planning; fill rates
History: Received: February 2014; Accepted: October 2014 by Chelliah Sriskandarajah, after 2 revisions.

1. Introduction

Consider the following scenario:

> Tablet computers in the iPad series include several different models based on different combinations of storage size and wireless chip technology. For expositional purposes, assume that an iPad can be assembled using three components: (a) Component 1 is the case color which is either White or Black; (b) Component 2 is the wireless chip Wi-Fi Only, or Wi-Fi together with 3G; and (c) Component 3 is the storage size which is either 16GB, 32GB, or 64GB. There are 12 distinct end-products, each equipped to deliver a specific level of functionality, that can be assembled based on the case color (component 1), wireless chip technology (component 2), and storage size (component 3).

This example reflects the trend toward the proliferation of product variants stemming from a need to capture niche customer segments and increase market share. Advances in production technologies have evened out differentials in product quality, and shifted the focus from mass production—with the virtues of low cost, efficiency, and consistent quality—to mass customization. Modularity is one of the keys to successful mass customization (Eggen 2003). As in mass production, mass customization achieves cost savings from the scale economies of repetitive manufacturing of each of the parts or modules that make up the modular end product. There may also be additional indirect sources of cost savings, such as greater reliability stemming from the use of common components that are reused over time and incrementally improved with each reuse, and reduced service costs and claim costs associated with modular new product introductions (Sanchez and Mahoney 2002). Modular product architectures also permit faster technological upgrading and greater speed to market, since an innovation in a single module has a multiplicative effect through the manifold potential combinations of each module with several others in creating the end-product. However, one should note that modularity also bounds the extent to which customization can be achieved and is not a substitute for pure customization. Further, a modularization strategy taken too far may result in brand cannibalization and customer confusion about the hierarchy of related product offerings; in such cases an integral product architecture is a better solution (Eggen 2003).

From an operational perspective, modularity creates unique challenges since a gamut of day-to-day decisions supporting product variety, ranging from manufacturing and service delivery to logistics, must be carefully planned. Ramdas (2003) points out that high variety can "increase demand variability and forecast errors" leading to the classic syndrome of market mismatch: cycles of excess inventory alternating with cycles of shortage. Determining the optimal stocks for a large number of the individual end-product variants (12 in the opening example in the present paper) is not simple since the aggregate product demand (in this case, for the iPad) and the preferences of consumers for each component option (e.g., the storage options—16GB, 32GB, or 64GB) are both uncertain parameters whose estimates are subject to forecast error. Ramdas (2003) also points to demand forecasting in the context of the combinatorial explosion of product variety resulting from modularization as an important untapped research area.

In this paper, we study the problem of determining optimal stocks of the end-product variants in a single product family from the perspective of a retailer or distributor. Shelf-space in retail stores is scarce, and it is vital for a retailer to right-size inventory investment without compromising service levels. The focus of our research is on determining optimal stocking levels for a suite of modular products in the face of two distinct sources of uncertainty: random aggregate product demand spanning all possible modular combinations, and unknown market preferences for various options at the level of an individual module. These two sources of uncertainty are pinpointed by Ramdas (2003), who states that "when introducing a new product category there is uncertainty about aggregate demand for the new category, as well as the demand for specific varieties within the category." Our model maximizes expected profit subject to two classes of service level constraints: an aggregate end-product fill rate constraint, as well as individual variant-specific fill rate constraints. Our most detailed results are for a single period model, but we carry out an extension to an infinite horizon model with complete backlogging and non-zero replenishment lead time. The infinite horizon model may be used to make stocking decisions over a finite rolling horizon with positive replenishment lead time.

The remainder of this paper is structured as follows. In the next section, we give a brief overview of the relevant literature and position our paper with respect to it. Section 3 presents the general single-period model and the associated analytics, and applies it to the iPad series assembly stocking problem. In section 4 using an extensive numerical analysis, we provide insights about the interplay between modularity, market preferences, and service levels in planning a stocking assortment. Section 5 presents an extension of the benchmark model including correlated preference over options, price-dependent demand, alternative probability distributions for characterizing uncertainty in market preferences or aggregate demand, and an infinite horizon setting with positive lead time. In section 6, we briefly reprise the main contributions of this paper and point out some of the limitations of our analysis.

2. Literature Review and Positioning

Our paper has points of contact with three distinct streams of literature, but it also deviates from the prototypical papers in each stream in some essential features. We shall attempt to carefully define the boundaries of each relevant literature stream so that the positioning of our paper against this backdrop is clearly marked out.

First, we consider the setting and objectives of a stream of research on inventory policies for stochastic assembly systems. The objective of research in this category is to determine the optimal stocks of the components of a modular product subject to random aggregate demand. For a given realization of demand for a set of products, the problem of assigning components to end-products is in principle a stand-alone deterministic optimization problem. However, when demand is random and component stocks have to be planned before demand realizes, the difficulty of determining optimal component stocks *a priori* escalates significantly and generally takes the form of a two-stage stochastic optimization problem with recourse. For example, Thomas and Warsing (2007) study a periodic, order-up-to system for a single product assembled from multiple components. In the first stage, stocking decisions must be made with respect to components and assembly. At the recourse stage, decisions must be made regarding assembly and disassembly actions and allocation of inventories to demands. Shortage costs are charged, rather than imposing service level constraints. Chod et al. (2010) is another noteworthy recent paper in this stream of research.

Our work intersects research on assemble-to-order (ATO) systems in that our focus is also on a set of modular products. However, in contrast to the cited literature, we are concerned with the stocks of end product variants that must be planned ahead of customer demand realizations, and thus, demand is satisfied later when it is realized (i.e., our setting is make-to-stock rather than assemble to order). Therefore, we are concerned with the individual modules making up the end-product only from the point of view of requiring a forecast of end-product demand. Frequently, it is the case that the demand for an end-product may be developed from information about consumer preferences for option choices at the level of a product module. However, we are not concerned with planning component stocks, or with optimally configuring specific sets of end-product variants from a given base of components. These are important problems that belong upstream in the supply chain.

Baker et al. (1986) present a simple model for an ATO setting to explore product commonality issues. Their product structure is somewhat similar to ours in the sense that there are two end products, each assembled from two components. The authors compare the case when each product has two unique components with the case in which the products each have one unique component and a common component. The demand for each product is assumed to be independent and uniform. The problem is to choose the component stock levels to minimize the total number of components stocked subject to an aggregate service level constraint, whereby the probability of meeting the demand of both products jointly is no smaller than an exogenously specified threshold value. In contrast, our model enjoins fill rate constraints rather than type-1 service level constraints; we impose individual as well as aggregate service level constraints; and our product is assembled using one option chosen for each component rather than the single option for each component in their setting.

Bertsimas and Paschalidis (2001) study optimal production and sequencing decisions a multiple-product make-to-stock manufacturing system, via a fluid model analyzed with tools from large deviations theory. Demand is met from the available finished goods inventory or backordered if there is a shortage. The objective is to find a production policy that minimizes finished goods inventory costs and guarantees that the steady-state stock-out probabilities for each product exceed a threshold.

A second stream of research pertains to the multiple-product newsvendor problem. We refer the reader to Turken et al. (2012) for a comprehensive review of research on this problem. Whereas the set up of the multiple-product newsvendor problem is a single period, Aviv and Federgruen (2001)—among other authors—study capacitated multiple-product inventory systems in an infinite horizon framework. Our paper resembles the multi-product newsvendor model in that we choose the stocking levels of a number of products each with random demand. However, the demands of the products for which we must make stocking decisions are inter-related by the underlying modular structure of the products. Furthermore, we impose multiple fill rate constraints whereas much of the research on the multi-product newsvendor model operates with budget constraints.

We also extend our single-period model to an infinite horizon, or equivalently, to a rolling horizon model in which the effect of non-zero replenishment lead time on optimal stocking levels can be studied. Of course, replenishment lead time is not modeled in the multi-product newsvendor model. Van Mieghem and Rudi (2002) introduced the notion of newsvendor networks, a class of models that generalize the standard newsvendor model by allowing for multiple products and multiple inputs with an arbitrary bill-of-materials structure. These models result in stochastic programming problems with recourse, and in this sense they may be viewed as an extension of ATO models. Bish et al. (2012) models a newsvendor network problem in which capacity levels must be determined under uncertain demand; once demand realizes, prices and production levels must be set. Unlike Van Mieghem and Rudi (2002), prices are endogenous.

A third stream of research pertains to the study of service-level constrained inventory models. The literature has classified inventory service levels into the following three categories: α-service-level, β-service-level and γ-service-level. α-service-level, also called Type 1 service level, is the fraction of cycles in which a stockout does not occur. A stockout occurs when demand arrives and there is no inventory available to satisfy that demand immediately. There exists a variant of α-service-level, which is called ready rate. Ready rate denotes the probability that an arbitrarily arriving customer order will be completely served from stock. β-service-level, also called Type 2 service level and fill rate, denotes the expected fraction of demand served immediately from stock. We focus on studying fill rate through the rest of this paper since this is the metric that most managers use to measure service level (Nahmias 2008). γ-service-level is a less common service level but closely related to the fill rate. Schneider (1981) has provided an exceptional review of these three inventory service levels under different replenishment policies. Interested readers may also refer to Silver et al. (1998) who provide additional characterizations of these service level approaches with multiple illustrations for each setting.

Fill rate is consistently defined as the average fraction of demand that can be immediately satisfied from on-hand inventory (Cachon and Terwiesch 2008, Song 1998) in the operation management literature. Fill rate is often computed as expected demand satisfied per cycle, divided by expected demand per cycle (this formula is equivalent as one minus the expected back order per cycle divided by expected demand per cycle). However, Chen et al. (2003) first point out this formula only holds when the demand is stationary and serially independent over an infinite horizon. They go on to also prove that the expected fill rate over a finite performance review horizon with a fixed stocking quantity is always greater than the expected fill rate over an infinite performance review horizon. In a follow-up paper, Banerjee et al. (2005) generalize Chen et al.'s (2003) work by showing the expected fill rate is monotonically decreasing in the review horizon. Thomas (2005) uses Monte Carlo simulation to study how the achieved fill rate behaves over a range of different demand distributions and review horizons. Katok et al. (2008) further investigates this issue in a controlled laboratory setting and conclude that longer review periods is more effective than shorter ones at inducing service improvements.

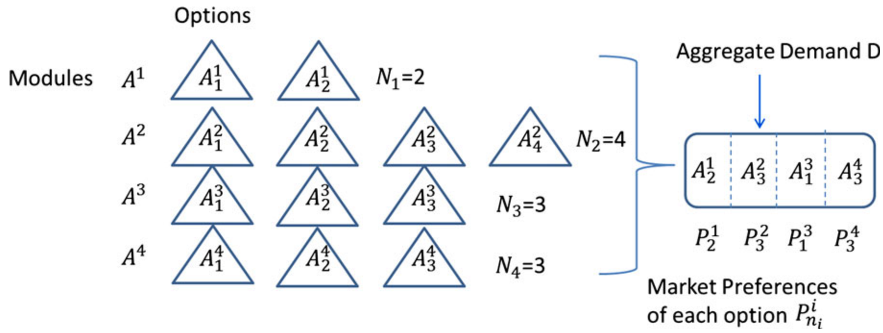
Several researchers have also studied fill rate in periodic review systems with positive lead time. Johnson et al. (1995) provide the details of the derivation of the classical fill rate formula and propose an exact fill rate expression with the normal distribution. Sobel (2004) derives a formula for fill rate under general demand distributions for both single-stage and multiple-stage supply chain systems. Zhang et al. (2007) extend Sobel's (2004) work to the general periodic review policy where the inventory position is reviewed once every R periods for single-stage and two-stage inventory systems. Teunter (2009) has derived the same expression for the fill rate in Zhang et al. (2007) by using an alternative approach. Recently, Guijarro et al. (2012) has developed a generalized method to compute the fill rate with lost sales and discrete demand distribution in a periodic review policy. The focus of all the above works is on characterizing the fill rate in a single or multistage inventory system over an infinite horizon.

Finally, we mention Paul and Vakharia (2006), where the focus is on determining optimal component levels to minimize total expected cost subject to a pre-set probability that all aggregate end-product demand is met (type-1 service level). The current model is different in that we maximize expected profit subject to individual fill rates for each product variant as well an aggregate fill rate for all the products together, motivated by two observations in particular: (a) fill-rates are far more common than type-one service levels in industry practice; and (b) it is vital to track service level at the *individual* product level when we make stocking decisions for a set of related product variants. Analytically, the stochastic dependence between numerator and denominator makes fill rate constraints particularly challenging in a stochastic inventory model. The change in service level measure changes the analytics of the problem drastically and calls for a completely different solution approach. Further, in Paul and Vakharia (2006) much effort is expended in exploring and proving structural results, many of which are limited to the special case of two options per component. The current paper takes a more comprehensive approach in the sense that industry cost and pricing data are used to generate extensive computational results under different scenarios, and we mine the output for insights on the interaction between demand variability, option choice, and individual and aggregate service levels.

3. Benchmark Model

We consider a family of product variants in a given modular product family, with random total demand D in a single period. Each individual product variant is assembled by choosing a specific option from each module; all the products are configured from the same set of modules and differ from one another in the chosen options from one or more modules. Each module is functionally a sub-assembly, but we prefer to use the term module here since we do not model the assembly process and the associated decision making. Let A^i ($i = 1, \ldots, M$) represent each module and suppose there are N_i ($n_i = 1, \ldots, N_i$) options for each module indexed as $A^i_{n_i}$. This implies that there are potentially $\prod_{i=1}^{M} N_i$ different end-product variants. The market preferences for customers for each option $A^i_{n_i}$ within module A^i ($i = 1, \ldots, M$) are random, and are represented by proportions $P^i_{n_i}$ (for $i = 1, \ldots, M$; $0 \leq P^i_{n_i} \leq 1$ and $\sum_{n_i=1}^{N_i} P^i_{n_i} = 1$). The market preferences are general random variables taking values in [0,1], and are assumed to be independent across modules in the benchmark model.[1] We use the term "market preferences" rather than "choice probabilities" because we want to connote that a market preference is an aggregate measure summarizing the choices of a large number of consumers in the market. Figure 1 illustrates a specific example where the market demand for product using options A^1_2, A^2_3, A^3_1 and A^4_3 is $DP^1_2P^2_3P^3_1P^4_3$, where D denotes the aggregate

Figure 1 An Example of Modules and Options in a Modular Production System



demand. Our approach of modeling consumer preferences over the levels, or variants, of an SKU attribute (such as storage size levels of an iPad in GB) is standard in the marketing research literature (e.g., Fader and Hardie 1996, where consumer preferences for SKU attributes of packaged consumer goods are modeled).

Let $c_{n_1,n_2,\ldots n_M}$ and $k_{n_1,n_2,\ldots n_M}$ represent the unit cost and unit price, respectively, of a product variant with options $A^1_{n_1}A^2_{n_2}\ldots A^M_{n_M}$. We enjoin an aggregate fill rate constraint for the entire product family as follows:

$$E\left[\frac{\text{\# of units of demand satisfied from stock}}{\text{\# of units of aggregate demand}}\right] \geq \beta \qquad (1)$$

and we also impose individual fill-rate constraints, one for each end-product variant:

$$E\left[\frac{\text{\# of units of demand satisfied from stock}}{\text{\# of units of specific product demand}}\right] \geq \beta_{n_1,n_2,\ldots n_M} \qquad (2)$$

Our decision variables are the stocking levels of each product variant $S_{n_1,n_2,\ldots n_M}$ ($\forall \prod_{i=1}^{M} N_i$ different end-product variants). The firm's profit maximization problem is as follows:

$$\text{Maximize SP} = E\left[\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\cdots\sum_{n_M=1}^{N_M} k_{n_1,n_2,\ldots n_M} \times Min(S_{n_1,n_2,\ldots n_M}, DP^1_{n_1}P^2_{n_2}\ldots P^M_{n_M})\right] - \sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\cdots\sum_{n_M=1}^{N_M} c_{n_1,n_2,\ldots n_M}S_{n_1,n_2,\ldots n_M} \qquad (3)$$

subject to:

$$E\left[\frac{\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\cdots\sum_{n_M=1}^{N_M} Min(S_{n_1,n_2,\ldots n_M}, DP^1_{n_1}P^2_{n_2}\ldots P^M_{n_M})}{D}\right] \geq \beta, \qquad (4)$$

$$E\left[\frac{Min(S_{n_1,n_2,\ldots n_M}, DP^1_{n_1}P^2_{n_2}\ldots P^M_{n_M})}{DP^1_{n_1}P^2_{n_2}\ldots P^M_{n_M}}\right] \geq \beta_{n_1,n_2,\ldots n_M}, \quad \forall n_1, n_2, \ldots n_M \qquad (5)$$

$$S_{n_1,n_2,\ldots n_M} \geq 0. \qquad \forall n_1, n_2, \ldots n_M \qquad (6)$$

The objective function (Equation 3) assesses the net profit for the firm, with the first term capturing the expected revenue from the sale of all product variants and the second term representing the procurement cost. The first constraint (Equation 4) requires that a minimum aggregate fill rate β be met. There are $\prod_{i=1}^{M} N_i$ individual fill rate constraints, one for each product variant, specifying the minimum expected fill rates required for each product. Each set of minimum fill rates is associated with a specific optimal profit. The fill rate constraints are critical in our setting because fill rates are important to retailers or distributors, although they are difficult to measure accurately (see Sobel (2004) for many examples of retailers emphasizing the importance of maintaining high fill rates). Further, these fill rate constraints obviate the need to include goodwill costs, which are theoretically appealing but hard to ascribe a dollar value to.

3.1. Random Preferences and Random Aggregate Demand

We start by first reducing this model to a tractable algebraic form. The preferences for module i are represented by N_i distinct random variables, each taking values in [0,1], and adding up to 1. We generate these N_i random variables as follows. First we generate $N_i - 1$ independent and identically distributed random variables $U_1, \ldots, U_{N_i-1}$ each with support [0,1]. Let $(V_1, \ldots, V_{N_i-1})$ denote the order statistics obtained by arranging the U_i's in ascending order. Define $V_0 := 0$ and $V_{N_i} = 1$. The random variables $D_i = V_i - V_{i-1}$ for $1 \leq i \leq N_i$ are said to be the *spacings* from the random sample $U_1, \ldots, U_{N_i-1}$. We represent market preferences for modular options by spacings. We remark that with this model every realization of market preferences results in different probabilities across modular options, since the probabilities are generated by negatively correlated continuous random variables. In this sense, the spacings model of market preferences across modular options captures heterogeneous market preferences across the variants of every module.

Pyke (1965) is a classic source for technical details regarding spacings. The distribution function of each spacing from a sample of n independent [0,1] random variables can be characterized. Based on this, the distribution function of $P^1_{n_1}P^2_{n_2}\dots P^M_{n_M}$—which we denote by $G_{n_1,n_2,\dots,n_M}(x)$—and the distribution function of $DP^1_{n_1}P^2_{n_2}\dots P^M_{n_M}$—which we denote by $H_{n_1,n_2,\dots,n_M}(x)$—can be found, if the distribution of aggregate demand N is known. We denote $1 - F(x)$ by $\overline{F}(x)$, for any distribution function $F(x)$. Then the objective function

$$E\left[\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\dots\sum_{n_M=1}^{N_M}k_{n_1,n_2,\dots n_M}\text{Min}(S_{n_1,n_2,\dots n_M},DP^1_{n_1}P^2_{n_2}\dots P^M_{n_M})\right] -\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\dots\sum_{n_M=1}^{N_M}c_{n_1,n_2,\dots n_M}S_{n_1,n_2,\dots n_M}$$

reduces to

$$\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\dots\sum_{n_M=1}^{N_M}k_{n_1,n_2,\dots n_M}\int_0^{S_{n_1,n_2,\dots n_M}}\overline{H}_{n_1,n_2,\dots,n_M}(x)dx -\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\dots\sum_{n_M=1}^{N_M}c_{n_1,n_2,\dots n_M}S_{n_1,n_2,\dots n_M} \quad (7)$$

Next, we derive an expression for aggregate fill rate. Our method is to first find the conditional expectation of aggregate fill rate given $D = n$, and then uncondition by integrating with respect to the distribution of D. Let $g_D(x)$ denote the density function of aggregate demand, with support $[A,B]$. The conditional expectation of aggregate fill rate given $D = n$ is

$$E\left[\frac{\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\dots\sum_{n_M=1}^{N_M}\text{Min}(S_{n_1,n_2,\dots n_M},nP^1_{n_1}P^2_{n_2}\dots P^M_{n_M})}{n}\right]$$

which reduces to

$$\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\dots\sum_{n_M=1}^{N_M}\frac{\int_0^{S_{n_1,n_2,\dots n_M}}\overline{H}'_{n_1,n_2,\dots,n_M}(x)dx}{n}$$

where H' indicates the distribution function of $nP^1_{n_1}P^2_{n_2}\dots P^M_{n_M}$. Now we uncondition, to obtain the following expression for the aggregate fill rate:

$$\int_A^B\left[\frac{g_D(y)}{y}\right]\left[\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\dots\sum_{n_M=1}^{N_M}\int_0^{S_{n_1,n_2,\dots n_M}}\overline{H}_{n_1,n_2,\dots,n_M}(x)dx\right]dy \quad (8)$$

Finally, we derive a formula for individual expected fill rate. We define

$$X_{n_1,\dots,n_M} := DP^1_{n_1}P^2_{n_2}\dots P^M_{n_M}$$

and let $H_{n_1,\dots,n_M}(x)$, $h_{n_1,\dots,n_M}(x)$ and denote its distribution function and density function, respectively. We note that each individual expected fill rate

$$E\left[\frac{\text{Min}(S_{n_1,n_2,\dots n_M},X_{n_1,\dots,n_M})}{X_{n_1,\dots,n_M}}\right]$$

can be simplified via conditioning to take the form

$$E\left[\frac{\text{Min}(S_{n_1,n_2,\dots n_M},X_{n_1,\dots,n_M})}{X_{n_1,\dots,n_M}}|X_{n_1,\dots,n_M}<S_{n_1,n_2,\dots n_M}\right] Prob(X_{n_1,\dots,n_M}<S_{n_1,n_2,\dots n_M}) + E\left[\frac{\text{Min}(S_{n_1,n_2,\dots n_M},X_{n_1,\dots,n_M})}{X_{n_1,\dots,n_M}}|X_{n_1,\dots,n_M}\geq S_{n_1,n_2,\dots n_M}\right] Prob(X_{n_1,\dots,n_M}\geq S_{n_1,n_2,\dots n_M})$$

which further simplifies to

$$\begin{aligned}&H_{n_1,\dots,n_M}(S_{n_1,n_2,\dots n_M}) + E\left[\frac{S_{n_1,n_2,\dots n_M}}{X_{n_1,\dots,n_M}}|X_{n_1,\dots,n_M}\geq S_{n_1,n_2,\dots n_M}\right]\overline{H}_{n_1,\dots,n_M}(S_{n_1,n_2,\dots n_M})\\ &= H_{n_1,\dots,n_M}(S_{n_1,n_2,\dots n_M}) + \int_{S_{n_1,n_2,\dots n_M}}^{\infty}\frac{S_{n_1,n_2,\dots n_M}}{x}\frac{h_{n_1,\dots,n_M}(x)}{\overline{H}_{n_1,\dots,n_M}(S_{n_1,n_2,\dots n_M})}dx\overline{H}_{n_1,\dots,n_M}(S_{n_1,n_2,\dots n_M})\\ &= H_{n_1,\dots,n_M}(S_{n_1,n_2,\dots n_M}) + S_{n_1,n_2,\dots n_M}\int_{S_{n_1,n_2,\dots n_M}}^{\overline{N}}\frac{h_{n_1,\dots,n_M}(x)}{x}dx\end{aligned} \quad (9)$$

This general model is intractable. In the next two sub-sections, we provide explicit algebraic formulations of the optimization problem for two special cases—fixed aggregate demand coupled with random market preferences, and fixed market preferences together with random aggregate demand. We study these two special cases in detail. In the rest of this study, we adopt the uniform distribution as our illustrating example for the following reasons. First, it captures the situation where the firm has no prior information of consumers' preferences for options and thus, it would be the worst case forecast scenario. Second, the directionality of our results still holds when we assume any other unimodal distribution for consumers preferences for options (as we show in a later section) and hence, this is not a very restrictive assumption. Third, we do not propose any market preference elicitation/estimation mechanism in our study since this is not the focus of our paper. Instead we simply assume that any such information on preferences is such that these preferences are uniformly distributed. Finally, the uniform distribution is also appealing owing to its tractability.

3.2. Fixed Aggregate Demand and Random Market Preferences

For ease of exposition, assume that each product in the family is assembled using options from 3 modules (i.e., $M = 3$), and there are two option choices available for modules 1 (i.e., A_1^1 and A_2^1) and 2 (i.e., A_1^2 and A_2^2), while there are three option choices are available for module 3 (i.e., A_1^3, A_2^3, and A_3^3). If the market preferences P_1^1 (and P_1^2) are uniformly distributed (0,1) random variables, then it follows that $P_2^1 = 1 - P_1^1$ ($P_2^2 = 1 - P_1^2$) are also uniformly distributed random variables. We model the market preferences for A_1^3, A_2^3, and A_3^3 as the spacings from three independent uniform [0,1] random variables. Since we are assuming in this case that D is known (that is, the firm has a fairly good forecast of the aggregate demand of the entire product family), we let $D = \overline{N}$.

3.2.1. Objective function. We need to derive an expression for the expected value of each term

$$k_{n_1,n_2,\ldots n_M}\mathrm{Min}\left(S_{n_1,n_2,\ldots n_M}, \overline{N}P^1_{n_1}P^2_{n_2}\ldots P^M_{n_M}\right)$$

and then sum up all the terms. We obtain the following expression for $E[\mathrm{Min}(S_{n_1,n_2,n_3}, \overline{N}P^1_{n_1}P^2_{n_2}P^3_{n_3})]$ (the details of the derivation are in the Appendix):

$$\frac{S_{n_1,n_2,n_3}}{12\overline{N}^2}\Big[12\overline{N}^2 - 15\overline{N}S_{n_1,n_2,n_3} + 4S^2_{n_1,n_2,n_3} -6\overline{N}S_{n_1,n_2,n_3}\ln\frac{S_{n_1,n_2,n_3}}{\overline{N}}(\ln\frac{1}{\overline{N}} + \ln S_{n_1,n_2,n_3} - 1)\Big]$$

Substituting into Equation (7), we have an explicit expression for the objective function.

3.2.2. Aggregate fill rate. The aggregate fill rate constraint can be written as

$$\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\cdots\sum_{n_M=1}^{N_M}\int_0^{S_{n_1,n_2,\ldots n_M}}\overline{H}'_{n_1,n_2,\ldots n_M}(x)dx \geq \beta\overline{N} \quad (10)$$

where $H'_{n_1,n_2,\ldots,n_M}(x)$ denotes the distribution function of $DP^1_{n_1}P^2_{n_2}\ldots P^M_{n_M}$. We use the expression already derived for $H'_{n_1,n_2,\ldots n_M}(x)$ in the Appendix and substitute the resulting expression for $\overline{H}'_{n_1,n_2,\ldots n_M}(x)$ into expression (8). After some rather laborious algebraic manipulation, the aggregate fill rate can finally be restated in the form:

$$\sum_{n_1=1}^{2}\sum_{n_2=1}^{2}\sum_{n_3=1}^{3}\frac{S_{n_1,n_2,n_3}}{12\overline{N}^3}\Big[12\overline{N}^2 - 15\overline{N}S_{n_1,n_2,n_3} -6\overline{N}S_{n_1,n_2,n_3}(\ln\frac{1}{\overline{N}} + \ln S_{n_1,n_2,n_3} - 1)\ln\frac{S_{n_1,n_2,n_3}}{\overline{N}} + 4S^2_{n_1,n_2,n_3}\Big]$$

3.2.3. Individual fill rates. Let $Z := DP^1_{n_1}P^2_{n_2}\ldots P^M_{n_M}$. Specializing the expression derived in the general case we get

$$\begin{aligned}E\left[\frac{\mathrm{Min}(S_{n_1,n_2,n_3},Z)}{Z}\right] &= H'_{n_1,n_2,\ldots,n_M}(S_{n_1,n_2,n_3}) + E\left[\frac{S_{n_1,n_2,n_3}}{Z}|Z \geq S_{n_1,n_2,n_3}\right]\overline{H}'_{n_1,n_2,n_3}(S_{n_1,n_2,n_3})\\ &= H'_{n_1,n_2,\ldots,n_M}(S_{n_1,n_2,n_3}) + S_{n_1,n_2,n_3}\int_{S_{n_1,n_2,n_3}}^{\overline{N}'}\frac{h'_{n_1,n_2,n_3}(x)}{x}dx\\ &= \frac{S_{n_1,n_2,n_3}\left[3S_{n_1,n_2,n_3} + 6\overline{N}\ln\overline{N} - \overline{N}(6\ln S_{n_1,n_2,n_3} + \ln^3\frac{S_{n_1,n_2,n_3}}{\overline{N}})\right]}{3\overline{N}^2}\end{aligned} \quad (11)$$

3.3. Fixed Market Preferences and Random Aggregate Demand

Let $P^1_{n_1} = p^1_{n_1}$, $P^2_{n_2} = p^2_{n_2}$, $\ldots$, $P^M_{n_M} = p^M_{n_M}$. Then to achieve a minimum aggregate fill-rate β the constraint in Equation (4) can be stated as:

$$E\left[\frac{\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\cdots\sum_{n_M=1}^{N_M}\mathrm{Min}(S_{n_1,n_2,\ldots n_M}, Dp^1_{n_1}p^2_{n_2}\cdots p^M_{n_M})}{D}\right] \geq \beta \quad (12)$$

We make an extensive study of the case where aggregate market demand D is uniformly distributed over $[0,\overline{N}]$.[2] The support of $\overline{N}p^1_{n_1}p^2_{n_2}\ldots p^M_{n_M}$ is $[0, \overline{N}p^1_{n_1}p^2_{n_2}\ldots p^M_{n_M}]$. After some algebraic drudgery—the details of which we omit—the aggregate fill rate constraint (4) reduces to

$$\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\cdots\sum_{n_M=1}^{N_M}\frac{S_{n_1,n_2,\ldots,n_M}}{\overline{N}}\left(1 + \ln\frac{\overline{N}p^1_{n_1}p^2_{n_2}\ldots p^M_{n_M}}{S_{n_1,n_2,\ldots,n_M}}\right) \geq \beta \quad (13)$$

Following the same method we used to derive the individual fill rate constraints in the previous sub-section, we simplify Equation (5) as follows:

$$\frac{S_{n_1,n_2,\ldots,n_M}}{\overline{N}p^1_{n_1}p^2_{n_2}\cdots p^M_{n_M}}\left(1+\ln\frac{\overline{N}p^1_{n_1}p^2_{n_2}\cdots p^M_{n_M}}{S_{n_1,n_2,\ldots,n_M}}\right)\geq\beta_{n_1,n_2,\ldots,n_M}\quad\forall n_1,n_2,\ldots,n_M \tag{14}$$

We again assume that the product family is assembled using options from three modules (i.e., $M = 3$). There are two option choices available for modules 1 (i.e., A^1_1 and A^1_2 with $p^1_1 + p^1_2 = 1$) and 2 (i.e., A^2_1 and A^2_2 with $p^2_1 + p^2_2 = 1$) while three option choices are available for module 3 (i.e., A^3_1, A^3_2, and A^3_3 with $p^3_1 + p^3_2 + p^3_3 = 1$). Then if we assume that D is uniformly distributed in the interval [0,200] (i.e., the mean aggregate demand for the product family is 100), Equation (13) simplifies to:

$$\sum_{n_1=1}^{2}\sum_{n_2=1}^{2}\sum_{n_3=1}^{3}\frac{S_{n_1,n_2,n_3}}{200}\left(1+\ln\frac{200p^1_{n_1}p^2_{n_2}p^3_{n_3}}{S_{n_1,n_2,n_3}}\right)\geq\beta\quad\forall n_1,n_2,n_3 \tag{15}$$

We simplify Equation (14) as

$$\frac{S_{n_1,n_2,n_3}}{200p^1_{n_1}p^2_{n_2}p^3_{n_3}}+\frac{S_{n_1,n_2,n_3}}{200p^1_{n_1}p^2_{n_2}p^3_{n_3}}(\ln 200p^1_{n_1}p^2_{n_2}p^3_{n_3}-\ln S_{n_1,n_2,n_3})\geq\beta_{n_1,n_2,n_3}\quad\forall n_1,n_2,n_3 \tag{16}$$

We get the following explicit expression for the objective function under fixed market preference and random aggregate market demand:

$$\sum_{n_1=1}^{2}\sum_{n_2=1}^{2}\sum_{n_3=1}^{3}k_{n_1,n_2,n_3}\left(S_{n_1,n_2,n_3}-\frac{S^2_{n_1,n_2,n_3}}{400p^1_{n_1}p^2_{n_2}p^3_{n_3}}\right)-\sum_{n_1=1}^{2}\sum_{n_2=1}^{2}\sum_{n_3=1}^{3}c_{n_1,n_2,n_3}S_{n_1,n_2,n_3} \tag{17}$$

We note the following structural results relating the aggregate fill rate constraint to the individual fill rate constraints, under symmetric and asymmetric market preferences.

Proposition 1. *When market preferences for the options in a given module are all equal (that is, preferences are symmetric across module options), the arithmetic mean of expected individual fill rates of the end-product variants is equal to the expected aggregate fill rate. On the other hand, when market preferences over the options are unequal (that is, preferences are asymmetric across module options), the expected aggregate fill rate is a convex combination of individual expected fill rates.*

Proof. See the Appendix. □

Fill rate is one of the most popular service levels to measure the performance of inventory system by practitioners. In practice, inventory managers are often confronted with a need to consider the aggregate fill rate constraint (De Schrijver et al. 2013). The above proposition suggests that one can achieve a good approximation of aggregate fill rate by focusing on the individual item fill rate when they have a good estimate on consumer preference over different functionalities of modular products. It is also noteworthy to point out that this proposition easily generalizes to the case when there are arbitrarily many modules, and an arbitrary number of options in each module.

Suppose we have very little information on consumer preferences for option. In this case, it might be obvious that the expected aggregate fill rate should be equal to some convex combination of expected individual fill rates. However, our later analysis demonstrates that this apparently mandatory relationship is violated when market preferences are random. Essentially, the expected aggregate fill rate is lower than the convex combination of expected individual fill rate.

In next section, we solve the algebraic models for the iPad series in the motivating example with the aid of the GAMS software package, over a wide range of problem parameters and with price and cost data obtained from secondary sources. The focus of our numerical analysis is to obtain more insights into how the parameter settings moderate the results for each case.

4. General Insights from Numerical Analysis

The results of the numerical analysis presented in this section are focused around comparing the differences in profits, total stocking levels, imputed aggregate and individual fill rates for two specific cases: (a) market preferences are random and aggregate demand (D) is Known; and (b) market preferences are known and aggregate demand (D) is random. In order to operationalize the algebraic formulations for each of these cases shown in the previous section, the cost and pricing data for the iPad series example were obtained from ISuppli (Rassweiler 2010). A summary of these data is shown in Table 1.[3] It is interesting to note that the same iPad data were also used by Subramanian et al. (2013) to conduct numerical experiments on their component commonality model. They made the following caveat: "Although our choice of iPads as an example may not be a perfect match with our model assumptions ... the publicly available component breakdown and cost data ... allows us to demonstrate how our model can be used .... " We would like to emphasize that the same caveat applies to our data too.

Table 1 IPAD Pricing and Cost Data

Product variant	Sale price ($)	Total cost ($)	Profit ($)	Margin %
W,WO,16GB	499.00	229.35	269.65	54.0
W,WO,32GB	599.00	258.85	340.15	56.8
W,WO,64GB	699.00	317.85	381.15	54.5
W,3G,16GB	629.00	257.65	371.35	59.0
W,3G,32GB	729.00	287.15	441.85	60.6
W,3G,16GB	829.00	346.15	482.85	58.2
B,WO,16GB	499.00	229.35	269.65	54.0
B,WO,32GB	599.00	258.85	340.15	56.8
B,WO,64GB	699.00	317.85	381.15	54.5
B,3G,16GB	629.00	257.65	371.35	59.0
B,3G,32GB	729.00	287.15	441.85	60.6
B,3G,16GB	829.00	346.15	482.85	58.2

The parameter settings in terms of consumer preferences and aggregate demand for each case are as follows:

- For the case of known D and random market preferences: We arbitrarily set the value of $D = \overline{N} = 100$ and assumed that the market preferences for options of components 1 and 2 were uniformly distributed random variables in the interval [0,1] while the market preferences of the options of component 3 as uniform spacing random variables in the interval [0,1].
- For the case of known market preferences and random D: We experiment with different values for the preferences of modular options, ranging from the perfectly balanced case (for components 1 and 2, each option preference= $\frac{1}{2}$); and for component 3, each option preference = $\frac{1}{3}$) to an unbalanced situation where preferences are price-sensitive and are inversely proportional to price. Further, we assume that D is uniformly distributed with parameters [0,200].

Finally, the input parameters for the minimum aggregate fill rate and minimum product variant fill rates are varied for each set of results discussed next.

Observation 1. *Optimal Profits.*

To compare optimal profits, we consider two specific settings for the minimum aggregate fill rate and minimum product variant fill rates. The first setting is where these parameters are set to be very small such that these constraints are never binding at optimum while in the second case, we set the minimum product variant fill rates to be 70% and vary the minimum aggregate fill rate. The results are presented below in Table 2 and Figure 2 and in obtaining these results preferences are set to be symmetric across options.

As can be seen from Figure 2, when the fill rate constraints (aggregate and individual) are not binding, the profits are approximately 2.5 times higher for the case when market preferences are known and D is random as compared to the case when market preferences are random and D is known. A more pronounced difference in profits can be observed with the minimum fill rate is set to be higher (in this case, all minimum individual fill rates were set to 70% and the minimum aggregate fill rate was varied in the range 40–70%) as seen in Table 2. For example, given individual fill rates of 70% and an aggregate fill rate of 60%, Table 2 shows that the profits are about 3.8 times higher for the case when market preferences are known and D is random as compared to the case when market preferences are random and D is known. Of course, greater profits are also accompanied by higher stocks of product variants but the difference in stocking levels for the two cases in not as pronounced as the differences in profits. This result implies that if the firm can obtain better estimates of consumer preferences for options, there is the potential to increase profits significantly (even if aggregate demand is random).

The forecasting of consumers preferences for options is a challenging task and this task is compounded when the number of options offered is very large. From a practical perspective, this implies that there is a need to provide some balance between the marketing focus to provide *more* consumer choice through options against the operational issue of preparing better accurate forecasts for such an increased option set.

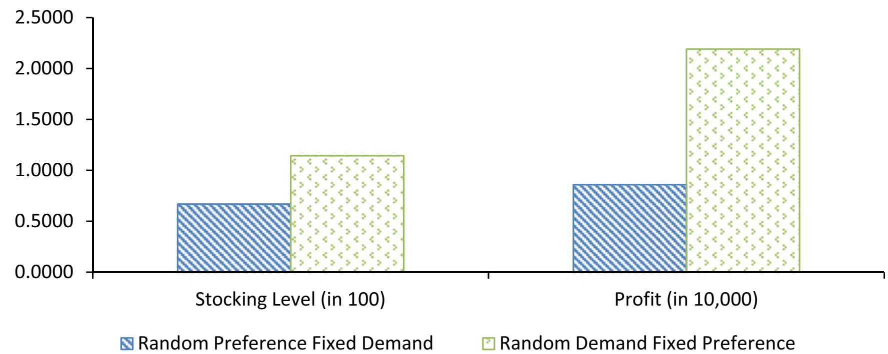
Observation 2. *Imputed Aggregate and Product Variant Fill Rates.*

Through Proposition 1, we know that when market preferences are known and aggregate demand is random, the aggregate fill rate is a weighted average of the individual product variant fill rates. However, this is not the case when market preferences are

Table 2 Profits Comparisons When Product Variant Fill Rates are Set at 70%

Aggregate fill rate	40%	45%	50%	55%	60%	65%	70%
Random Market Preference Fixed N	8,604.347	8,511.438	8,058.46	7,162.098	5,725.084	3,612.285	628.293
Random N, Known Market Preference	21,916.43	21,916.43	21,916.43	21,916.43	21,916.43	21,916.43	21,916.43

Figure 2 Comparison of Stocking Levels and Profit When Constraints are not Binding



random and aggregate demand is fixed. In order to illustrate the results for this case, we conducted a set of numerical experiments and the results are shown in Figure 3. The results are generated by fixing the individual product variant fill rate at 80% and varying the aggregate fill rate in the range of 40–80%. For each parameter setting, we tracked the imputed aggregate fill rates and product variant fill rates when optimizing profits.[4]

It is obvious that across the set of results presented in Figure 3, the imputed expected aggregate fill rate is strictly less than the *minimum expected product variant fill rate*. This result can be explained analytically as follows. Consider the 3 module product structure that was the focus of the previous section. The expected aggregate fill rate takes the form

$$\frac{\sum_i \sum_j \sum_k E\left[\text{Min}(S_{ijk}, \bar{N}P_{ijk})\right]}{\bar{N}} \tag{18}$$

where P_{ijk} and S_{ijk} are the succinct forms of $P^1_{n_1}P^2_{n_2}P^3_{n_3}$ and S_{n_1,n_2,n_3}, respectively. The individual fill rate for product S_{ijk} takes the form

$$E\left[\frac{\text{Min}(S_{ijk}, \bar{N}P_{ijk})}{\bar{N}P_{ijk}}\right]. \tag{19}$$

There are 12 terms in the sum in expression (18) for aggregate fill rate. To show the crux of the argument in the simplest possible manner, first consider the special case when the unit prices of the product variants are the same, and so are the unit costs. Then by symmetry, $S_{ijk} = S$ for all triples i, j, k and all the random variables $P_{ijk} \sim P$ have the same distribution. So we can then write the aggregate fill rate as

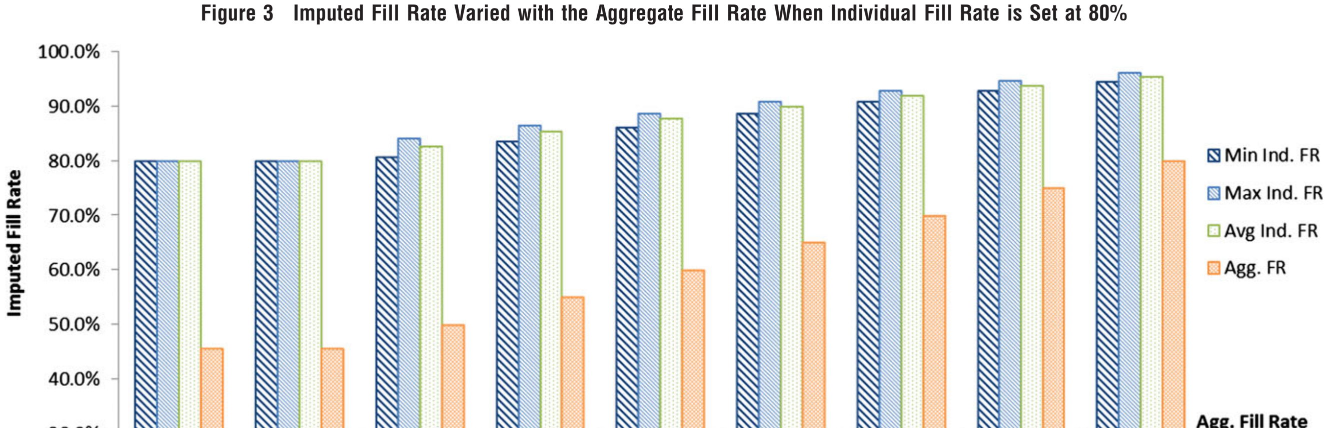
$$12\frac{E\left[\text{Min}(S, \bar{N}P)\right]}{\bar{N}} = \frac{E\left[\text{Min}(S, \bar{N}P)\right]}{\bar{N}/12} \tag{20}$$

and each individual fill rate as

$$E\left[\frac{\text{Min}(S, \bar{N}P)}{\bar{N}P}\right]. \tag{21}$$

Now it follows from results in Chen et al. (2010) and Banerjee and Paul (2005) that

Figure 3 Imputed Fill Rate Varied with the Aggregate Fill Rate When Individual Fill Rate is Set at 80%



	40%	45%	50%	55%	60%	65%	70%	75%	80%
Min Ind. FR	80.0%	80.0%	80.7%	83.6%	86.2%	88.7%	90.9%	92.8%	94.6%
Max Ind. FR	80.0%	80.0%	84.2%	86.5%	88.8%	90.9%	92.9%	94.6%	96.2%
Avg Ind. FR	80.0%	80.0%	82.6%	85.4%	87.8%	90.0%	92.0%	93.8%	95.4%
Agg. FR	45.6%	45.6%	50.0%	55.0%	60.0%	65.0%	70.0%	75.0%	80.0%

$$E\left[\frac{\text{Min}(S,\bar{N}P)}{\bar{N}P}\right] \geq \frac{E[\text{Min}(S,\bar{N}P)]}{\bar{N}EP} = \frac{E[\text{Min}(S,\bar{N}P)]}{\bar{N}/12} \tag{22}$$

Thus, in the special case when all the modules have identical costs and prices and preferences are symmetric, the aggregate fill rate is smaller than each individual fill rate. In fact, the difference can be quite significant. The main driver of this difference is the fact that $1/P$—which influences the individual fill rates but not the aggregate fill rate—is a very volatile random variable. Specifically, it can be verified that $1/P$ has a long-tailed distribution. The long-tailed behavior of $1/P$ stretches out the tail of the random variable $\frac{Min(S,\bar{N}P)}{\bar{N}P}$ and inflates expected individual fill rates relative to the short-tailed aggregate fill rate.

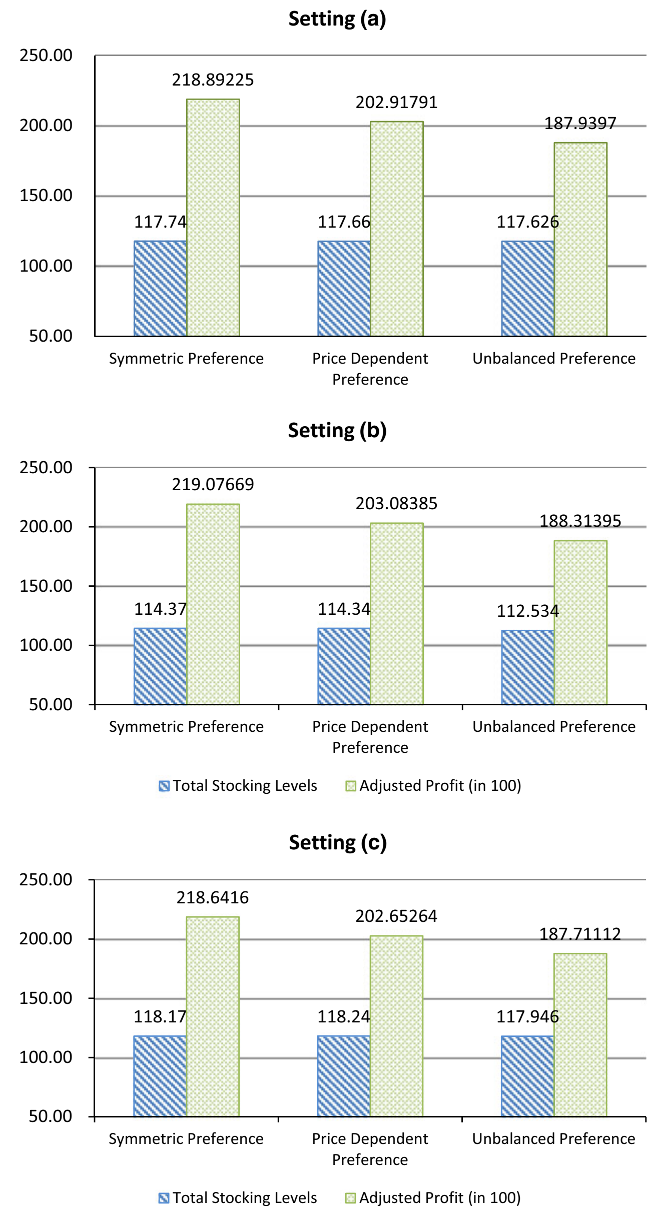
Observation 3. *Structure of Market Preferences.*

A final observation relates to how the structure of market preferences influences the stocking levels and optimal profits. To examine this issue, we focus on a setting where market preferences are known and aggregate demand is random. In this case, we allow the market preferences to be symmetric (i.e., equal across all options); price dependent (i.e., inversely proportional to price); and arbitrary (i.e., unbalanced across the options). For the price-dependent setting, we only vary the preferences for the storage options since this is the main factor which contributes to a pricing variations for the IPAD's. Hence, for this preference structure, we set $p_1^3 = 0.60$; $p_2^3 = 0.30$; and $p_3^3 = 0.10$. In the unbalanced case, we set the market preferences to be $p_1^1 = 0.30$; $p_2^1 = 0.70$; $p_1^2 = 0.70$; $p_2^2 = 0.30$; $p_1^3 = 0.60$; $p_2^3 = 0.30$; and $p_3^3 = 0.10$.

For each of these three settings of the preference structure, we generate three sets of results depending upon the minimum aggregate fill rate and minimum individual fill rates: (a) the minimum aggregate fill rate is set at 90% while the minimum individual fill rate is set at 20%; (b) the minimum aggregate fill rate is set at 70% while the minimum individual fill rate is set at 40%; and (c) the minimum aggregate fill rate is set at 20% while the minimum individual fill rate is set at 90%. The detailed results for stocking levels and optimal profits for each of these parameter settings are shown in Figure 4 below.

The results shown in Figure 4 indicate that there is very little difference between the total product stocks across the three market preference structures regardless of the type of preference structure and the settings for the parameters related to the minimum aggregate and individual fill rates. A symmetric market preference structure, on the other hand, leads to the greatest profits followed by when market preferences are inversely proportional to price and this result obviously holds across all parameter settings for the minimum aggregate and individual fill rates. Since symmetric preferences could be construed as providing less information than asymmetric preferences,[5] this result is somewhat counter-intuitive. On the other hand, symmetric preferences lead to balanced demands and balanced stocks across product options. Typically, it is easier to manage a homogeneous assortment of products than a heterogeneous one and balanced assortments may be expected to lead to higher profits. This may counteract the

Figure 4 Impact of Asymmetric Preference



negative informational aspect of the symmetric preferences model.

This concludes a discussion of our numerical experiments and in the next section, we discuss several extensions to our benchmark model.

5. Extended Models

We now consider extensions of the benchmark model to further assess the robustness of our key findings and to show how the single period model can be extended in a multiple-period setting. Without loss of generality, we investigate a stylized product setting where a single product family is assembled using two modules A^1 and A^2. One unit of module A^1 and one unit of module A^2 combine to form one unit of end product. There are two options for module A^1, denoted by A^1_1 and A^1_2, and two options for module A^2, denoted A^2_1 and A^2_2. So there are four possible variants of the end product, which we can represent as $A^1_1A^2_1$, $A^1_1A^2_2$, $A^1_2A^2_1$, and $A^1_2A^2_2$. The analysis in this section demonstrates that the value of information on preference continues to dominate the value of information on aggregate demand when the preference over options are correlated, when demand is price dependent with substitution and when demand and options follow unimodal distributions.

5.1. Correlated Preferences for Options

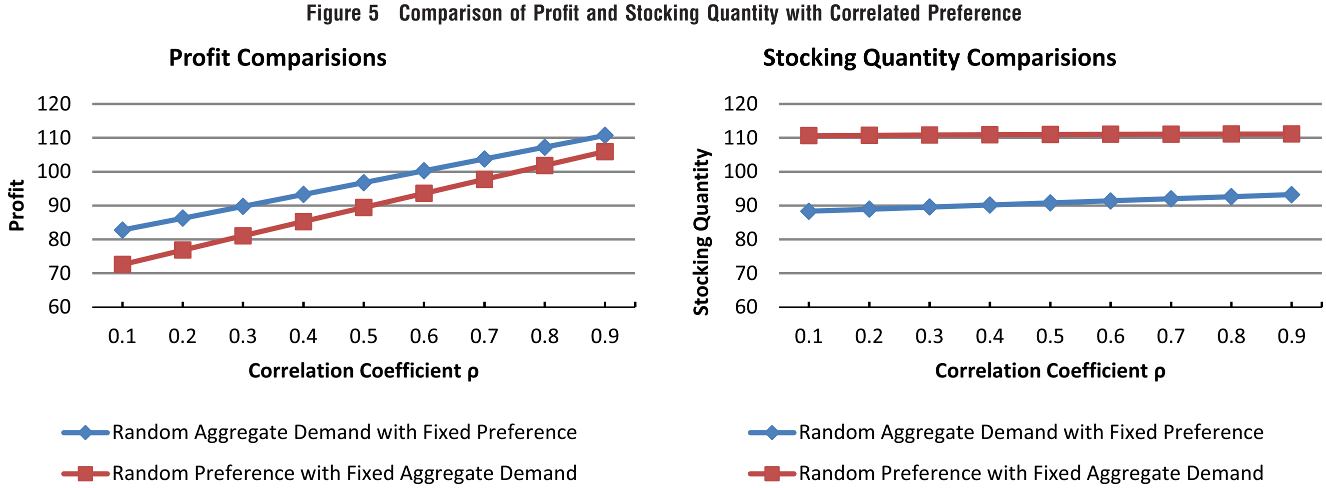
Consumers could have correlated preferences over options for the modular product. For example, it is reasonable to assume that a customer who favors 64GB storage size is more likely to choose 3G with Wi-Fi over Wi-Fi only as the wireless chip technology when choosing an iPad. We describe a stylized model to capture the correlation between customer preferences across options. Suppose each module consists of two options, $A^1 \in \{A^1_1 = A, A^1_2 = a\}$ and $A^2 \in \{A^2_1 = B, A^2_2 = b\}$, where the capital letter stands for the higher quality option. The market preferences for higher quality options are denoted by P^1 and P^2, respectively. Without loss of generality, we assume P^1 and P^2 follow the same distribution P taking values in the interval (0,1). Correlations between the preferences for the higher quality options in the two modules are induced by imposing the constraints

$$P(A^1_1 = A|A^2_1 = B) = P(A^2_1 = B|A^1_1 = A) \\ = P(A^1_2 = a|A^2_2 = b) = P(A^2_2 = b|A^1_2 = a) = \rho,$$

where $0 < \rho < 1$. Note that the conditional probability of choosing a higher quality option for a feature changes depending on the quality level of the option that is selected for the other feature; it is this that induces a correlation between option preferences across product features. The proportion of choosing product $\{\{A,B\},\{A,b\},\{a,B\},\{a,b\}\}$ can be seen to be equal to $(P\rho, P(1-\rho), (1-P)(1-\rho), (1-P)\rho)$, respectively. In the optimization model, we use the proportion derived here to replace the product form of preference where we assume consumers' preference over options are independent in the basic model (Figure 5).

We observe that the expected profit increases monotonically with ρ in both the random demand and the random preference cases, while the stocking levels increase slightly. To assure ourselves of the robustness of this effect, we constructed and tested another model of correlated preferences. In this

Figure 5 Comparison of Profit and Stocking Quantity with Correlated Preference



Notes: The results above are robust to the changes of parameters. For illustration purpose, we set the price parameters as follows: $\{k_{11} = 7, k_{12} = 5, k_{21} = 5, k_{22} = 4\}$ and the corresponding costs are: $\{c_{11} = 4, c_{12} = 3, c_{21} = 2, c_{22} = 1\}$. For the case of random preference with fixed aggregate demand, we set N = 100 and assumed the market preferences are uniformly distributed. In the case of random demand with fixed preference, we let N be uniformly distributed with parameters [0,200] and P = 1/2 model, we allowed the preferences between option 1 of modules 1 and 2, and between option 2 of modules 1 and 2, to be *correlated uniformly distributed random variables*. We note that in the correlation model constructed earlier correlation was induced by explicitly manipulating conditional probabilities of choosing an option of one feature given an option choice of another feature; here we model correlation at the level of the joint distribution of the preferences for a pair of options spanning two distinct modules—the qualitative idea is the same but the mathematical levers used are different in the two models of correlation. As we increased the coefficient of correlation, optimal stocking levels as well as optimal expected profits, again increased. This is in contrast to the general rule proved by Van Mieghem and Rudi (2002) for newsvendor networks characterized by multivariate normal demands whereby "the value of the system decreases in correlation" (p. 328) and demonstrates that there are factors at play in our model that go beyond conventional risk-pooling effects.

In our case, the intuition is that the sum of the variances of the product variant demands increases compared with the case where preferences are independent across modules, generating the need for more inventory to attain the same service level. To see that the sum of variances of product demands increases, note that $Var(Y_1 + \ldots + Y_n) = \sum_{i=1}^{n} Var(Y_i) + \sum_{i,j} Cov(Y_i, Y_j)$, where Y_i are product variant demands. Now the marginal distributions of Y_i remain unchanged in our correlation model, and hence the variance terms remain unchanged; hence the change in $Var(Y_1 + \ldots + Y_n)$ is driven entirely by the covariance terms.

The fact that optimal profits also rise, however, is somewhat counterintuitive since one would expect the impact of an increase in operational variability to be detrimental. For example, it is well known that an increase in variance—other things remaining the same—results in a decrease in expected profit in the newsvendor model, for large classes of demand distributions (see Van Mieghem and Rudi (2002), Proposition 3, for a generalization of the detrimental impact of variability on the objective function value in a newsvendor network model. However, the case when correlation increases when individual variance terms are unchanged is somewhat different. It is well known that in the case of a joint distribution, zero correlation corresponds to maximum entropy or minimum information, compared with the case of positive correlation. Therefore, an increase in correlation may be interpreted as an increase in information which may be expected to have some positive impact—at least when compared with the baseline case of zero correlation. The intuition that an increase in correlation should be detrimental is very model specific. An example of a setting in which an increase in correlation is beneficial may be found in project management. In a parallel project with stochastic activity times, project completion time is stochastically decreasing in the correlation between every activity pair, given that the activity durations are multivariate normal.

5.2. Price-Dependent Demand and Demand Substitution

In the benchmark model, we assume the aggregate demand is identical across the modular product family. Now we extend the benchmark model to include the following features:

(1) We incorporate price-dependent demand, with a downward sloping demand curve.
(2) We model cross-price demand substitution.

As before, let c_{n_1,n_2} and k_{n_1,n_2} represent the unit cost and unit price, respectively, of a product variant with options $A^1_{n_1}$ and $A^2_{n_2}$. We shall let the demand for each product variant conform to the multiplicative price-dependent random demand model (see Huang et al. 2013) with stock-out-based demand substitution (see Bish et al. 2012).

Specifically, we let the demand of product variant $A^1_{n_1}A^2_{n_2}$ be $\hat{D}_{n_1,n_2} * P^1_{n_1}P^2_{n_2}$ where $P^k_{n_j}$ are random variables taking values in (0,1) and summing to 1, and $\hat{D}_{n_1,n_2}$ is given by the following formula:

$$\hat{D}_{n_1,n_2} = \bar{D} - \alpha_{n_1,n_2}k_{n_1,n_2} + \sum_{m_1=1}^{2}\sum_{m_2=1}^{2}\beta_{m_1,m_2}k_{m_1,m_2} + \epsilon \quad (23)$$

where $m_i \neq n_i$ for all i. The first two terms in the formula for $\hat{D}_{n_1,n_2}$ capture a downward sloping linear demand curve for each product, where α_{n_1,n_2} reflects the own-price sensitivity. The third term captures demand substitution via cross price effects—the demand for a given product is increasing in the price of all the *other* products, as in Bish et al. (2012). The parameter $\beta_{m_1,m_2} > 0$ reflects the substitutability between the product variants where a higher value of β_{m_1,m_2} implies a higher substitutability effect between the product variant and vice versa. We impose the condition $\sum_{m_1=1}^{2}\sum_{m_2=1}^{2}\beta_{m_1,m_2} \leq \alpha_{n_1,n_2}$ to ensure the impact of own price change is larger than the impact of competing product price change. The random component $\varepsilon \sim K(\cdot)$ represents the randomness in demand with its mean equals to zero.

Now to obtain an explicit algebraic formulation for this model, we use an approximate formula for expected aggregate fill rate that is common in the operations management literature—that is Expected Fill Rate = (Expected Filled Demand)/(Expected

Demand). With this simplification, the aggregate fill rate constraint in the extended model is specified by

$$\frac{E\left[\sum_{n_1=1}^{2}\sum_{n_2=1}^{2}\text{Min}(S_{n_1,n_2},\hat{D}_{n_1,n_2}P_{n_1}^{1}P_{n_2}^{2})\right]}{E[\sum_{n_1=1}^{2}\sum_{n_2=1}^{2}\hat{D}_{n_1,n_2}P_{n_1}^{1}P_{n_2}^{2}]}\geq\beta \quad (24)$$

In both the random demand and random preferences models we observed that the average profit increased monotonically and linearly with $\beta_{i,j}$ keeping all the other parameters fixed. On the other hand average profit decreased monotonically and linearly with $\alpha_{i,j}$ keeping all the other parameters fixed. The slopes of the graph of average profit against $\beta_{i,j}$ were nearly identical in both random demand and random preferences cases, whereas the decrease in average profit with $\alpha_{i,j}$ was considerably steeper in the case of random preferences, compared with the case of random demand. Both these observed effects are, of course, in complete accord with intuition. We also found that the average optimal profit in the case when demand was random and preferences fixed dominated the average optimal profit in the case when demand was fixed and preferences random, as in the benchmark model. We reiterate the implication of this finding: market research targeted at eliciting consumer preferences for product features is in general a much more powerful lever than market research aimed at characterizing aggregate demand.

5.3. Unimodal Distributions

We now extend our analysis to more general classes of probability distribution. It is interesting to analyze stocking level when preferences and aggregate demand are unimodally distributed, capturing a situation when the firm possesses some information about relative preference and market demand, in contrast to the case of uniform distribution, which captures a situation when the firm has very little information.

We start our analysis when the preference is unimodally distributed and the aggregate demand is known. A random variable with support [0, 1] and continuous distribution function $F(x)$ is defined to be unimodal if there is a point v in [0, 1] such that $F(x)$ is convex on $[0,v]$ and concave on $[v,1]$. Differentiating the distribution function yields the density function, so a random variable is unimodal if it has a density function that is increasing up to v and decreasing from that point onwards. Generally, v is called the vertex, or the mode, of the distribution. Now for a very right-skewed density function, the vertex tends to lie very close to 0 and the random variable has a distribution function that is approximately concave. On the other hand, if the density function is very left-skewed, the vertex tends to be very near 1 and the distribution function is approximately convex. So from this point of view, concave and convex distribution functions represent the limiting distribution functions for right-skewed and left-skewed random variables, respectively. To provide a concrete example, we assume $D = \overline{N} = 100$ and model the market preferences $P_1^1$and P_1^2 with density functions as $g_{P_1^1}(\cdot) = 2 - 2x(0 \leq x \leq 1)$ and $g_{P_1^2}(\cdot) = 2x(0 \leq x \leq 1)$, respectively, which results in $P_2^1 = (1 - P_1^1)$ and $P_2^2 = (1 - P_1^2)$ being distributed as $g_{1-P_1^1}(\cdot) = 2x(0 \leq x \leq 1)$ and $g_{1-P_1^2}(\cdot) = 2 - 2x(0 \leq x \leq 1)$, respectively. Thus, we are able to fully operationalize the objective function and constraints.

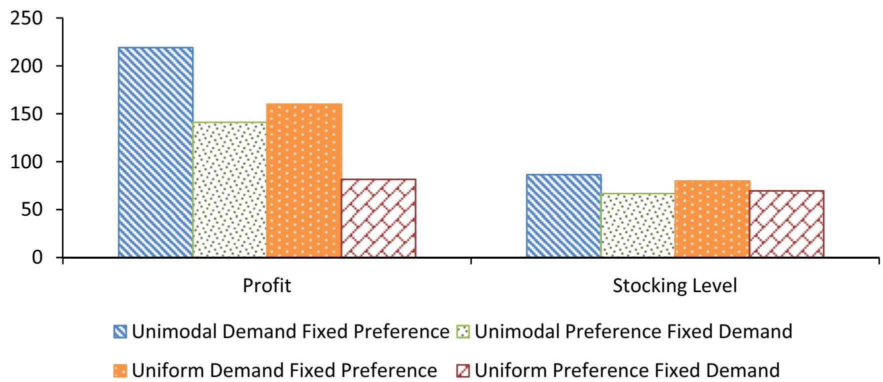
Now we consider the case where the firm forms a reliable estimate of the most likely value of market demand (i.e., mode of the market demand distribution) with known preferences. We study the corresponding change in optimal stocking level, compared with the case when demand is uniformly distributed. For convenience, let us consider the specific case where N has the unimodal density function $g_D(\cdot) = -\frac{3x(x-200)}{4{,}000{,}000}(0 \leq x \leq 200)$ with mode 100. We assume the preference over options equal to $P_1^1 = P_1^2 = \frac{1}{2}$, such that both cases share the same expected value of market demand. The results from the numerical experiment are summarized in the following Figure 6.

We briefly describe the most interesting inferences to drawn from our numerical experiments. To begin with, focusing on the comparison between unimodal demand with fixed preference and unimodal preference with fixed demand, we find our main qualitative insight still holds. That is, optimal profits increase substantially (i.e., the profits are about 1.5 times higher) when preferences are fixed rather than unimodally distributed. Further, we find a considerable increase in optimal profit as well as a reduction in optimal stock levels compared with the case of uniformly distributed preferences. Given that unimodal market preferences convey more information than uniform market proportions, this extra information can be judiciously used to reap savings in procurement costs. Managerially speaking, this result suggests that the company could save substantial costs by getting customer preferences information.

5.4. Infinite Horizon Model

We extend the benchmark model to a stationary infinite horizon setting. The expected fill rate is the expected fraction of demand served immediately from on-hand inventory, averaged over the infinite horizon. We call the time between two successive ordering opportunities in the horizon a period. The options ordered in period t are available to satisfy market demand in period $t + L$, where the lead time L is a non-negative integer number of periods. The

Figure 6 Comparison of Profit and Stocking Quantity with Different Randomness



Notes: The results above are robust to the changes of parameters. To focus on the effect of randomness in demand and preference, we set price and cost parameters equal across product variants, which equals to 10 and 6 respectively. And we set both aggregate and individual fill rates at 60%. For both cases involving fixed demand, we set N = 100

company is assumed to follow a base-stock policy for each modular product $A^1_{n_1}A^2_{n_2}$ with order-up-to level S_{n_1,n_2}.

The chronology of events is as follows: at the beginning of each period, replenishment shipments are received, backlogged demand is filled to the maximum possible extent from existing stock, then independent and identically distributed (i.i.d.) demand occurs during the period which will be satisfied from the available inventory or else backlogged, and finally a replenishment order is placed to bring next period's beginning inventory level up to S_{n_1,n_2}.

We introduce a unit holding cost h and a unit backlogging cost b. The objective function is to maximize the total discounted expected profit over the infinite horizon subject to an aggregate fill rate constraint and individual fill rate constraints over the infinite horizon. Our aim is to show that there is a myopic optimal solution to the infinite horizon problem. To this, we first note that the problem with the fill rate constraints deleted reduces to a multi-product newsvendor problem over the infinite horizon. It is a standard result (see Theorem 4.7 (p. 100) and the discussion on pp. 99 and 100 of Snyder and Shen (2011)) that the optimal solution to the single-product newsvendor problem over the infinite horizon is a stationary base-stock policy. Since the objective function in our problem is separable in the stock levels of the individual products, it follows that the optimal solution to the unconstrained version of our multi-period problem over the infinite horizon is a stationary base-stock policy. For this reason, it is reasonable to search for the optimal solution to the constrained problem in the class of stationary base-stock policies. To find the optimal stationary base stock policy for the constrained problem, we need to simplify the expected infinite horizon fill rate constraint.

Let H_{n_1,n_2} denote the on-hand inventory level of product $A^1_{n_1}A^2_{n_2}$ at the start of period t, and let $D_tP^1_{n_1}P^2_{n_2}$ denote the total demand for product $A^1_{n_1}A^2_{n_2}$ during period t. We assume aggregate stationary demand over the horizon, so $D_t = D$ for all t. Note that the stocking level S_{n_1,n_2} in the constraints in the single period model is now replaced by the on-hand inventory level H_{n_1,n_2}. We begin with the following characterization of expected fill rate over the infinite horizon.

PROPOSITION 2.

(a) *Satisfying the individual fill rate constraint over the infinite horizon reduces to* $\frac{E\left[\text{Min}(H_{n_1,n_2}, DP^1_{n_1}P^2_{n_2})\right]}{E[DP^1_{n_1}P^2_{n_2}]} \geq \beta_{n_1,n_2}$.

(b) *Satisfying the aggregate fill rate constraint over the infinite horizon reduces to* $\sum_{n_1=1}^{2}\sum_{n_2=1}^{2} E\left[\text{Min}(H_{n_1,n_2}, Dp^1_{n_1}p^2_{n_2})\right] \geq \beta E[D]$.

PROOF. Proof is provided in the Appendix. □

Note that the random variables H_{n_1,n_2} are identical and independent from period to period, by assumption; the same fact applies to aggregate demand D. Therefore, Proposition 2 implies that the fill rate constraints over the infinite horizon reduce to satisfying a set of identical constraints in every period of the horizon; let us call these constraints "modified fill rate constraints" since they take the form $\frac{EY}{EX} \geq \beta$ which is not equivalent to a single period fill rate constraint $E(Y/X) \geq \beta$. We note that these modified fill rate constraints can be satisfied by a suitable stationary policy. Hence the optimal stationary policy for the constrained problem can be solved by implementing the following procedure. First compute the optimal base-stock level for each product in the unconstrained problem. If these stock levels satisfy the modified fill rate constraints, then the stock levels are optimal. If not, escalate each stock level until all the modified fill rate constraints are satisfied. The resulting stock levels constitute a stationary optimal solution to the constrained problem.

We can state the single period problem as follows (we set the backlogging cost to zero since we have imposed fill rate constraints, which represent imputed shortage costs).

$$\text{Maximize SP} = \sum_{n_1=1}^{2}\sum_{n_2=1}^{2}\left[(k_{n_1,n_2} - c_{n_1,n_2})E(Dp^1_{n_1}p^2_{n_2})\right] - \frac{1}{2}h\sum_{n_1=1}^{2}\sum_{n_2=1}^{2}k_{n_1,n_2}E[H_{n_1,n_2}] \quad (25)$$

subject to the fill rate constraints

$$E\left[\text{Min}(H_{n_1,n_2}, Dp^1_{n_1}p^2_{n_2})\right] \geq \beta_{n_1,n_2}E\left[Dp^1_{n_1}p^2_{n_2})\right] \quad \forall n_1, n_2 \quad (26)$$

$$\sum_{n_1=1}^{2}\sum_{n_2=1}^{2}E\left[\text{Min}(H_{n_1,n_2}, Dp^1_{n_1}p^2_{n_2})\right] \geq \beta E[D] \quad (27)$$

Since we assume complete backlogging, the expected revenue per period is fixed (being equal to the expected demand multiplied by the margin per product) and the objective function reduces to minimizing expected holding cost per period:

$$\text{Minimize SP} = \frac{1}{2}h\sum_{n_1=1}^{2}\sum_{n_2=1}^{2}k_{n_1,n_2}E[H_{n_1,n_2}] \quad (28)$$

To solve this problem, we use the result in Proposition 3 according to which we need to compute the distribution of $H_{n_1,n_2} = \max(S_{n_1,n_2} - \Lambda_{n_1,n_2}, 0)$, where Λ_{n_1,n_2} denotes the total lead time demand of product $A^1_{n_1}A^2_{n_2}$.

When the lead time is positive, the on-hand inventory at the start of each period is a random variable. The following proposition characterizes this random variable.

PROPOSITION 3. *The on-hand inventory level at any review instant for a product with order-up-to level S and lead time demand Λ is the random variable H = max(S − Λ,0).*

PROOF. Proof is provided in the Appendix. □

To apply this result to our problem, we need to compute the distribution of $Y = \max(S_{n_1,n_2} - \Lambda_{n_1,n_2}, X)$ where X is any random variable independent of S_{n_1,n_2} and Λ_{n_1,n_2}. We shall consider the case of a demand distribution with a finite support, since we conduct our numerical sensitivity analysis for uniformly distributed demand. So the lead time demand for each end-product variant has a finite upper bound $\bar{\lambda}$. There are two cases to consider.

CASE 1: $\bar{\lambda} \leq S_{n_1,n_2}$. *We have*

$$\begin{aligned}\text{Prob}(Y \leq t) &= \text{Prob}(S_{n_1,n_2} - \Lambda_{n_1,n_2} \leq t)\text{Prob}(X \leq t) \\ &= \text{Prob}(\Lambda_{n_1,n_2} \geq S_{n_1,n_2} - t)\text{Prob}(X \leq t).\end{aligned}$$

We need to reduce this to the case when $X = 0$ with probability 1. Making the substitution, we get the distribution function of H_{n_1,n_2}:

$$\text{Prob}(H_{n_1,n_2} \leq t) = \text{Prob}(\Lambda_{n_1,n_2} \geq S_{n_1,n_2} - t) \quad (29)$$

Now we have

$$\begin{aligned}&\text{Prob}(\text{Min}(H_{n_1,n_2}, Dp^1_{n_1}p^2_{n_2}) \geq t) \\ &= \text{Prob}(H_{n_1,n_2} \geq t)\text{Prob}(Dp^1_{n_1}p^2_{n_2}) \geq t \\ &= \text{Prob}(\Lambda_{n_1,n_2} \leq S_{n_1,n_2} - t)\text{Prob}(Dp^1_{n_1}p^2_{n_2}) \geq t.\end{aligned}$$

Denoting the distribution of Λ_{n_1,n_2} *by* $J(.)$, *we get*

$$E[\text{Min}(H_{n_1,n_2}, Dp^1_{n_1}p^2_{n_2})] = \int_0^{\bar{\lambda}} \overline{F}_{n_1,n_2}(x)J(S_{n_1,n_2} - x)dx \quad (30)$$

Note that $J(.)$ is the L-fold convolution of $Dp^1_{n_1}p^2_{n_2}$ and $\overline{F}_{n_1,n_2}$ denotes the distribution function of the demand for product $A^1_{n_1}A^2_{n_2}$ in one period.

CASE 2: $\bar{\lambda} > S_{n_1,n_2}$. *In this case, the same method as the one in Case 1 can be applied except that* $H_{n_1,n_2} = \max(S_{n_1,n_2} - \Lambda_{n_1,n_2}, 0)$ *takes a different form. In case, 1, we have* $H_{n_1,n_2} = \max(S_{n_1,n_2} - \Lambda_{n_1,n_2}, 0) = S_{n_1,n_2} - \Lambda_{n_1,n_2}$ *since* $S_{n_1,n_2} > \Lambda_{n_1,n_2}$ *with probability* 1.

We provide complete algebraic details for both cases for a concrete problem instance in the appendix. In both cases, we set lead time $L = 1$ and the results are summarized in Table 3. Optimal stocking levels are roughly twice that in the zero lead time single period model. This is what one would expect; it is well-known that the "vulnerable window" in a periodic review model with $L = 1$ period and period length $T = 1$ is $L + T = 2$ periods, which is twice that in a single period model. The results are shown in the above table. The variance of the observed results from a ratio of 2 is explained by the fact that the objective functions in the two models were somewhat different; the single period model incorporated lost sales in contrast to complete backlogging in the infinite horizon model. Finally, we note that for every fill rate level, the stocking quantity in the multiple period model is significantly higher with random preference than with random aggregate demand. Note that higher stocks directly lead to lower expected profit in the multiple period model. So we have once again verified the principle that accurate

Table 3 Comparison of Stocking between Single Period and Multi-Period

	Single period		Multi-period	
Fill rate (%)	Random preference	Random demand	Random preference	Random demand
60	100.80	119.34	204.00	186.65
65	103.63	122.89	221.80	196.67
70	117.18	135.74	241.05	206.90
75	132.54	147.52	262.25	218.29
80	150.36	158.62	268.20	231.31
85	171.78	169.27	314.32	246.74
90	199.14	179.67	349.71	266.11

information on consumer preferences (captured by fixed p in our model) pays greater dividends than accurate information about aggregate demand (captured by fixed N in our model.

6. Concluding Remarks

In this paper, we studied the problem of determining optimal stocks of the end-product variants in a single product family, with the aim of maximizing expected profit subject to fill rate constraints. We modeled two distinct sources of uncertainty: random aggregate product demand spanning all possible modular combinations, and unknown market preferences for various options at the level of an individual module. We applied our analytics to the case of tablet computers in the iPad series. We used industry component cost and end-product pricing data and hypothetical demand and market preference distributions to derive optimal base-stock levels for iPad end-product variants. We gleaned several insights from the sensitivity analysis, clarifying the interplay between the shape of market preferences for modular options and aggregate demand variability on the one hand, and optimal stocking levels, fill rates, and expected profits on the other. Through extensive computational analysis, we find that precise estimates of market preferences for various modular options constitute valuable information that goes beyond the usefulness of forecasts of aggregate market demand. From a practical perspective, this might be indicative of another classic marketing-operations trade-off. Offering more options for consumers would be preferred by marketing managers since this would reach more consumers and hence, enhance product sales. On the other hand, obtaining more accuracy in forecasts would decline when the number of options is larger. Hence, from an operational perspective, it would be preferred to limit option choices (so that better forecasts can be obtained) since this would lead to lower stocking costs and hence, higher profits.

We briefly sketch some ideas for future research suggested by our work in this paper. We focused for the most part on one source of uncertainty at a time: random aggregate demand, or random market preferences. We did obtain some idea of stocking levels and fill rates both aggregate demand and preferences were simultaneously random, via Monte Carlo simulation. We observed that optimal stocking levels were lower, and the optimal expected profit and fill rates were smaller, than when there was a single source of randomness. We modeled randomness in options preferences via random spacings from a uniform distribution, thereby imposing some measure of homogeneity on market preferences for the various options of a given module. A more explicitly heterogeneous model that we might compare the present results against would be one in which options preferences are elicited as random spacings from a skewed distribution on the unit interval. Our sensitivity analysis was based on optimizing with respect to uniformly distributed random demand; one might wonder whether any of our findings are tied to this distributional assumption. We did not tackle the finite horizon version of the problem, choosing to address the limitations of the single period model partway with the help of a tractable infinite horizon model in which we considered only the class of stationary base-stock policies. Situating the same problem in a finite horizon model—which one might argue is the true state of the world—would introduce serious complications and require new algorithms and computational procedures outside the repertoire of the present paper. Finally, we ignored stock-out-based substitution by customers. An example of recent research that models this feature is Honhon and Seshadri (2013), who analyze a model with consumer-driven substitution and random consumer preferences; although their model is quite different from ours, their focus on the contrast between fixed and random preferences or proportions is very similar to ours.

This paper adds to the rich vein of research into the management of modular products, a body of work spanning at least three distinct streams in the literature: components planning for ATO systems with the emphasis on upstream requirements planning, assortment planning for horizontally and vertically differentiated products with the focus on selecting an assortment of products from a large set of candidate products jostling for scarce retail shelf-space, and the many variations of the multiple-product newsvendor model.

Acknowledgments

The authors are grateful to the department editor, senior editor, and the review team for their thorough and thoughtful comments and suggestions, which greatly helped to improve the paper. Anand Paul is the corresponding author. This study is partially supported by a Warrington College of Business Administration Summer Research Grant awarded to Asoo J. Vakharia.

Appendix A. An Objective Function in Fixed Aggregate Demand and Random Market Preferences

Recall that we denoted the distribution function of $P^1_{n_1}P^2_{n_2}\dots P^M_{n_M}$ by $G_{n_1,n_2,\dots,n_M}(x)$ and the distribution function of $DP^1_{n_1}P^2_{n_2}\dots P^M_{n_M}$ by $H_{n_1,n_2,\dots,n_M}(x)$.

We know from the distribution theory of spacings that the distribution function of each uniform spacing random variable in the product $P^1_{n_1}P^2_{n_2}\dots P^M_{n_M}$ is of the form $1-(1-x)^{n-1}$ when there are n different modular options. This knowledge is sufficient to derive the distribution function $H'_{n_1,n_2,\dots,n_M}(x)$ of $\overline{N}P^1_{n_1}P^2_{n_2}\dots P^M_{n_M}$. Some manipulative algebraic work yields the following explicit expression for $H'_{n_1,n_2,n_3}(x)$:

$$H'_{n_1,n_2,n_3}(x)=\frac{x}{\overline{\overline{N}}}\left(-\frac{x}{\overline{\overline{N}}}+\ln^2\left(\frac{x}{\overline{\overline{N}}}\right)+2\right)\quad 0<x<\overline{N}$$

Observing that the random variable $\text{Min}(S_{n_1,n_2,n_3},DP^1_{n_1}P^2_{n_2}P^3_{n_3})$ takes only non-negative values, we use the formula of $E[X]=\int_0^\infty P(X\geq x)dx$ where $X\sim H(\cdot)$ to simplify the expression. After some calculation, this simplifies to

$$E\left[Min(S_{n_1,n_2,n_3},NP^1_{n_1}P^2_{n_2}P^3_{n_3})\right]=\int_0^\infty 1-H_{n_1,n_2,n_3}(x)dx$$
$$=\int_0^{S_{n_1,n_2,n_3}}1-\frac{x}{\overline{\overline{N}}}\left(-\frac{x}{\overline{\overline{N}}}+\ln^2\left(\frac{x}{\overline{\overline{N}}}\right)+2\right)dx$$

Appendix B. Derivation of Stylized Model with Uniform Distribution

To facilitate the readers to use our model in other research settings, we provide technical details on how to reformulate the optimization problem. For illustration purpose, we assume there is a single product family with four possible variants of the end product $A^1_1A^2_1$, $A^1_1A^2_2$, $A^1_2A^2_1$ and $A^1_2A^2_2$. One unit of module A^1 and one unit of module A^2 combine to form one unit of end product. There are two options for module A^1, denoted by A^1_1 and A^1_2, and two options for module A^2, denoted A^2_1 and A^2_2. One can solve a more general problem by following the similar techniques with evolving algebraic steps. Based on the above description, the firm's profit maximization problem is as follows:

$$\text{Maximize SP}=E\left[\sum_{n_1=1}^{2}\sum_{n_2=1}^{2}k_{n_1,n_2}\text{Min}(S_{n_1,n_2},DP^1_{n_1}P^2_{n_2})\right]-\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}c_{n_1,n_2}S_{n_1,n_2}$$

subject to:

$$E\left[\frac{\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}Min(S_{n_1,n_2},DP^1_{n_1}P^2_{n_2})}{D}\right]\geq\beta,$$
$$E\left[\frac{Min(S_{n_1,n_2},DP^1_{n_1}P^2_{n_2})}{DP^1_{n_1}P^2_{n_2}}\right]\geq\beta_{n_1,n_2},\quad\forall n_1,n_2$$
$$S_{n_1,n_2}\geq 0.\quad\forall n_1,n_2$$

We denote the distribution function of $DP^1_{n_1}P^2_{n_2}$ by $H_{n_1,n_2}(x)$. We first focus on the case of fixed aggregate demand with uniform preference over options. The market preferences of P^1_1 (and P^2_1) are uniformly distributed (0,1) random variables, then it follows that $P^1_2=1-P^1_1$ ($P^2_2=1-P^2_1$) are also uniformly distributed random variables. Thus, we can characterize $H_{n_1,n_2}(x)=\frac{x}{\overline{N}}(1-\ln\frac{x}{\overline{N}}), x\in[0,\overline{N}]$, where $\overline{N}$ represents the fixed aggregate demand. The objective function can be simplified as,

$$\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\left[k_{n_1,n_2}\int_0^{S_{n_1,n_2}}\overline{H}_{n_1,n_2}(x)dx-c_{n_1,n_2}S_{n_1,n_2}\right]$$
$$=\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\frac{k_{n_1,n_2}S_{n_1,n_2}(4\overline{N}-3S_{n_1,n_2}+2S_{n_1,n_2}\ln\frac{S_{n_1,n_2}}{\overline{N}})}{4\overline{N}}-c_{n_1,n_2}S_{n_1,n_2}$$

Similarly, we can simplify the aggregate fill rate constraint as,

$$\frac{E\left[\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}Min(S_{n_1,n_2},\overline{N}P^1_{n_1}P^2_{n_2})\right]}{\overline{N}}=\frac{\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}S_{n_1,n_2}(4\overline{N}-3S_{n_1,n_2}+2S_{n_1,n_2}\ln\frac{S_{n_1,n_2}}{\overline{N}})}{4\overline{N}^2}\geq\beta$$

For the individual fill rate, we resort to the technique that we develop in Equation (9),

$$E\left[\frac{\text{Min}(S_{n_1,n_2},NP^1_{n_1}P^2_{n_2})}{NP^1_{n_1}P^2_{n_2}}\right]$$
$$=H_{n_1,n_2}(S_{n_1,n_2})+S_{n_1,n_2}\int_{S_{n_1,n_2}}^{\overline{N}}\frac{h_{n_1,n_2}(x)}{x}dx$$
$$=\frac{S_{n_1,n_2}\left(2+\ln\frac{S_{n_1,n_2}}{\overline{N}}(\ln\frac{S_{n_1,n_2}}{\overline{N}}-2)\right)}{2\overline{N}}\geq\beta_{n_1,n_2}\quad\forall n_1,n_2$$

Thus, we fully operationalize the optimization problem. Next, we focus on the case when aggregate demand is uniformly distributed and the preferences over options are fixed. For convenience, we assume $P^1_1=P^2_1=\frac{1}{2}$ and $N\sim Uniform(0,\overline{N})$. We can characterize $G_N(x)=\frac{x}{\overline{N}}, x\in[0,\overline{N}]$ and $H_{n_1,n_2}(x)=\frac{4x}{\overline{N}}$,

$x \in [0, \frac{\overline{N}}{4}]$, respectively. Thus, the objective function can be simplified as,

$$\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\left[\int_0^{S_{n_1,n_2}} \overline{H}_{n_1,n_2}(x)dx - c_{n_1,n_2}S_{n_1,n_2}\right] = \sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\left(S_{n_1,n_2} - \frac{S_{n_1,n_2}^2}{100}\right) - c_{n_1,n_2}S_{n_1,n_2}$$

Next we simplify the aggregate fill rate constraint as,

$$E\left[\frac{\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\text{Min}(S_{n_1,n_2}, NP_{n_1}^1P_{n_2}^2)}{N}\right]$$
$$=\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\frac{S_{n_1,n_2}}{\overline{N}} + \int_{\frac{S_{n_1,n_2}}{\overline{N}}}^{\frac{1}{4}} 1 - \left[1 - G_N\left(\frac{S_{n_1,n_2}}{x}\right)\right][1 - G_N(4S_{n_1,n_2})]dx$$
$$=\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\frac{S_{n_1,n_2}}{\overline{N}} + \frac{(4\overline{N} - S_{n_1,n_2})\left(\overline{N} - 4S_{n_1,n_2} + 4S_{n_1,n_2}\ln\frac{4S_{n_1,n_2}}{\overline{N}}\right)}{16\overline{N}^2}$$
$$\geq \beta$$

For the individual fill rate, we again resort to the technique that we develop in Equation (9),

$$E\left[\frac{\text{Min}(S_{n_1,n_2}, DP_{n_1}^1P_{n_2}^2)}{DP_{n_1}^1P_{n_2}^2}\right]$$
$$= H_{n_1,n_2}(S_{n_1,n_2}) + S_{n_1,n_2}\int_{S_{n_1,n_2}}^{\frac{\overline{N}}{4}} \frac{h_{n_1,n_2}(x)}{x}dx$$
$$= \frac{S_{n_1,n_2}}{50} + \frac{4S_{n_1,n_2}\ln\frac{\overline{N}}{4S_{n_1,n_2}}}{\overline{N}} \geq \beta_{n_1,n_2} \quad \forall n_1, n_2$$

Thus, the simplification of objective function is complete

Appendix C. Proof of Proposition 1

The aggregate fill-rate constraint takes the form

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{3}\frac{S_{ijk}}{N}\left(1 + \ln\frac{Np_i^1p_j^2p_k^3}{S_{ijk}}\right) \geq \beta$$

while the individual fill rate constraints take the form

$$\frac{S_{ijk}}{Np_i^1p_j^2p_k^3}\left(1 + \ln\frac{Np_i^1p_j^2p_k^3}{S_{ijk}}\right) \geq \beta_{ijk}$$

for all triples (*i,j,k*) where *i,j* = 1,2, and *k* = 1,2,3. When preferences are equal we have $p_i^1p_j^2p_k^3 = \frac{1}{2}\frac{1}{2}\frac{1}{3} = \frac{1}{12}$ for all triples (*i,j,k*). Now it is easy to check that when we substitute $p_i^1p_j^2p_k^3 = \frac{1}{12}$ for each end-product variant S_{ijk} and take the average, we obtain the expression for aggregate fill rate with the same substitution $p_i^1p_j^2p_k^3 = \frac{1}{12}$. When preferences are unequal, the aggregate fill rate takes the form

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{3}p_i^1p_j^2p_k^3\theta_{ijk}$$

where θ_{ijk} is the fill rate attained by end-product variant S_{ijk}. Since $\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{3}p_i^1p_j^2p_k^3 = 1$ and $p_i^1 \geq 0$, $p_j^2 \geq 0$, and $p_k^3 \geq 0$ for all *i,j,k*, the claim in the unequal preferences case follows.

Appendix D. Proof of Proposition 2

(a) Each individual fill rate constraint over the infinite horizon is of the form $\lim_{T\to+\infty} E\left[\frac{\sum_{t=1}^{T}\text{Min}(H_{n_1,n_2}, D_tP_{n_1}^1P_{n_2}^2)}{\sum_{t=1}^{T}D_tP_{n_1}^1P_{n_2}^2}\right]$. It follows directly from the renewal reward theorem (Ross 1983, Theorem 3.6.1) that that $\lim_{T\to+\infty} E\left[\frac{\sum_{t=1}^{T}\text{Min}(H_{n_1,n_2}, D_tP_{n_1}^1P_{n_2}^2)}{\sum_{t=1}^{T}D_tP_{n_1}^1P_{n_2}^2}\right] = \frac{E[Min(H_{n_1,n_2}, DP_{n_1}^1P_{n_2}^2)]}{E[DP_{n_1}^1P_{n_2}^2]}$. This establishes the proposition.

(b) The proof is similar to that of (a).

Appendix E. Proof of Proposition 3

Consider a review instant *t*. The inventory position at *t* is *S*, and the inventory position at *t* + *L* is *S* as well. We claim that the on-hand inventory at *t* + *L* is *max*(*S* − Λ,0). To see this, note that everything on-order at time *t* will have arrived by time *t* + *L* and no order placed after *t* will have arrived at that point. Hence, if there were zero demand during *t* and *t* + *L*, the on-hand inventory at time *t* + *L* would be precisely *S*. In fact, the on-hand inventory must be reduced by the lead time demand Λ, and if the realization of Λ is larger than *S*, the on-hand inventory at time *t* + *L* will be driven to zero. Therefore, the distribution of the on-hand inventory level at any review instant is the distribution of the random variable *max* (*S* − Λ,0).

Appendix F. Infinite horizon model example

We provide complete algebraic details for the case when demand is random and preferences are fixed and assume: *L* = 1, $p_{n_1}^1 = p_{n_2}^2 = 1/2$ and *D* is uniformly distributed between 0 and 200. Similar steps can be used to characterize the results when demand is fixed and preferences are random, which we skip here. The price parameters are the same as in the previous sub-sections. Each of the four end-product variants therefore has demand uniformly distributed over [0,50]. We have two cases to consider.

CASE 1: $S_{n_1 n_2} > 50$. Let $Z = \text{Min}(H_{n_1,n_2}, Dp^1_{n_1}p^2_{n_2})$. Define $X = S_{n_1 n_2} - U_1$, and $Y = U_2$, where U_1 and U_2 are independent random variables each uniformly distributed over [0,50]. Then $X \sim H_{n_1,n_2}$ and $Y \sim Dp^1_{n_1}p^2_{n_2}$, and we have

$$\text{Prob}(Z \ge t) = \begin{cases} \text{Prob}(Y \ge t),\ 0 \le t \le S_{n_1 n_2} - 50; \\ \text{Prob}(X \ge t)\text{Prob}(Y \ge t)\ S_{n_1 n_2} - 50 \le t \le 50; \\ t > 50. \end{cases}$$

Letting $G_T(.)$ denote the distribution function of any random variable T, we have

$$\begin{aligned} E[\text{Min}(H_{n_1,n_2}, Dp^1_{n_1}p^2_{n_2})] &= \int_0^\infty \overline{G_Z}(t)dt \\ &= \int_0^{S_{n_1 n_2}-50} \text{Prob}(Y \ge t)dt \\ &\quad + \int_{S_{n_1 n_2}-50}^{50} \text{Prob}(X \ge t)\text{Prob}(Y \ge t)dt \\ &= \int_0^{S_{n_1 n_2}-50} \overline{G_Y}(t)dt + \int_{S_{n_1 n_2}-50}^{50} \overline{G_Y}(t)\overline{G_X}(t)dt \\ &= -\frac{125}{3} + 2S_{n_1 n_2} - \frac{S^2_{n_1 n_2}}{50} + \frac{S^3_{n_1 n_2}}{15000} \end{aligned}$$

CASE 2: $S_{n_1 n_2} \le 50$.

$$\text{Prob}(Z \ge t) = \begin{cases} \text{Prob}(X \ge t)\text{Prob}(Y \ge t), & 0 \le t \le S_{n_1 n_2}; \\ 0, & t \ge S_{n_1 n_2}. \end{cases}$$

So we get

$$\begin{aligned} E[\text{Min}(H_{n_1,n_2}, Dp^1_{n_1}p^2_{n_2})] &= \int_0^S \text{Prob}(X \ge t)\text{Prob}(Y \ge t)dt \\ &= \int_0^S [1 - \frac{50 - S_{n_1 n_2}}{50}][1 - \frac{t}{50}]dt \\ &= \frac{S^2_{n_1 n_2}}{100} - \frac{S^3_{n_1 n_2}}{15000} \end{aligned}$$

Therefore, the optimization problem is the following:

$$\text{Minimize } \frac{1}{2}h \sum_{n_1=1}^{2}\sum_{n_2=1}^{2} k_{n_1 n_2}(S_{n_1 n_2} - 25)$$

subject to

$$-\frac{125}{3} + 2S_{n_1 n_2} - \frac{S^2_{n_1 n_2}}{50} + \frac{S^3_{n_1 n_2}}{15000} \ge 25\beta_{n_1 n_2} \forall n_1, n_2$$

$$\sum_{n_1=1}^{2}\sum_{n_2=1}^{2} -\frac{125}{3} + 2S_{n_1 n_2} - \frac{S^2_{n_1 n_2}}{50} + \frac{S^3_{n_1 n_2}}{15000} \ge 100\beta$$

if $S_{n_1 n_2} \ge 50$ and

$$\text{Minimize } \frac{1}{2}h \sum_{n_1=1}^{2}\sum_{n_2=1}^{2} k_{n_1 n_2}\frac{S^2_{n_1 n_2}}{100}$$

subject to

$$\frac{S^2_{n_1 n_2}}{100} - \frac{S^3_{n_1 n_2}}{15000} \ge 25\beta_{n_1 n_2} \forall n_1, n_2$$

$$\sum_{n_1=1}^{2}\sum_{n_2=1}^{2} \frac{S^2_{n_1 n_2}}{100} - \frac{S^3_{n_1 n_2}}{15000} \ge 100\beta$$

if $S_{n_1 n_2} < 50$.

Notes

[1]This assumption is valid in a lot of scenarios. For example, one's preference of a specific color has nothing to do with the preference over other options. Later in the extended model, we show that our main results still hold with correlated market preferences.
[2]We could, of course, have chosen to specialize the general development of the previous subsection to any other tractable distribution.
[3]http://www.isuppli.com/Teardowns/News/Pages/Mid-RangeiPadtoGenerateMaximumProfitsforApple,iSuppliEstimates.aspx (accessed date October 20, 2011).
[4]Although we do not show the resulting optimal retailer profits for each experimental parameter setting, it is worth noting that the maximum profit is obtained when the imputed individual product variant fill rates equal 75–80% and the aggregate fill rate equals 41%. The drives the choice of generating the results in Figure 3 where we set the minimum individual product variant fill rates to be at least 80% and vary the minimum aggregate fill rate.
[5]This is based on the classical result from information theory that the uniform discrete distribution over a finite set has *greater entropy*—and therefore conveys less information—than any other discrete distribution over the same set.

References

Aviv, Y., A. Federgruen. 2001. Design for postponement: A comprehensive characterization of its benefits under unknown demand distributions. *Manage. Sci.* **49**(4): 578–598.

Baker, K. R., M. J. Magazine, H. Nuttle. 1986. The effect of commonality on safety stock in a simple inventory model. *Manage. Sci.* **32**(8): 982–988.

Banerjee, A., A. Paul. 2005. Average fill rate and horizon length. *Oper. Res. Lett.* **33**(5): 525–530.

Banerjee, A., A. Paul. 2008. Path correlation and PERT bias. *Eur. J. Oper. Res.* **189**(3): 1208–1216.

Bertsimas, D., I. Paschalidis. 2001. Probabilistic service level guarantees in make-to-stock manufacturing systems. *Oper. Res.* **49**(1): 119–133.

Bish, E. K., X. Zeng, J. Liu, D. Bish. 2012. Comparative statics analysis of multiproduct newsvendor networks under responsive pricing. *Oper. Res.* **50**(5): 1111–1124.

Cachon, G., C. Terwiesch. 2008. *Matching Supply with Demand: An Introduction to Operations Management*. McGraw-Hill/Irwin, Edition 2, New York.

Chen, J., D. K. Lin, D. Thomas. 2003. On the single item fill rate for a finite horizon. *Oper. Res. Lett.* **31**(2): 119–123.

Chen, Y., J. E. Carrillo, A. J. Vakharia, P. Sin. 2010. Fusion product planning: A market offering perspective. *Decision Sci.* **41**(2): 325–353.

Chod, J., D. Pyke, N. Rudi. 2010. The value of flexibility in make-to-order systems the effect of demand correlation. *Oper. Res.* **55**(4): 834–848.

Eggen, O. 2003. Modular Product Development, Department of Product Design. unpublished document. Norwegian University of Science and Technology.

Fader, P., B. Hardie. 1996. Modeling consumer choice among SKUs. *J. Market. Res.* **33**(4): 442–452.

Guijarro, E., M. Cardos, E. Babiloni. 2012. On the exact calculation of the fill rate in a periodic review inventory policy under discrete demand patterns. *Eur. J. Oper. Res.* **218**(2): 442–447.

Honhon, D., S. Seshadri. 2013. Fixed vs. random proportions demand models for the assortment planning problem under stockout-based substitution. *Manuf. Serv. Oper. Manag.* **15**(3): 378–386.

Huang, J., M. Leng, M. Parlar. 2013. Demand functions in decision modeling a comprehensive survey and research directions. *Decision Sci.* **44**(3): 557–609.

Johnson, E. M., H. L. Lee, T. Davis, R. Hall. 1995. Expressions for item fill rates in periodic inventory systems. *Nav. Res. Log.* **42**(1): 57–80.

Katok, E., D. Thomas, A. Davis. 2008. Inventory service-level agreements as coordination mechanisms: The effect of review periods. *Manuf. Serv. Oper. Manag.* **10**(4): 609–624.

Nahmias, S. 2008. *Production and Operations Analysis*. McGraw-Hill/Irwin, Edition 6, New York.

Paul, A., A. J. Vakharia. 2006. Requirements planning for modular products. *Nav. Res. Log.* **53**(5): 418–431.

Pyke, R. 1965. Spacings. *J. R. Stat. Soc. Ser. B* **27**(3): 395–449.

Ramdas, K. 2003. Managing product variety: An integrative review and research directions. *Prod. Oper. Manag.* **12**(1), March 2003, 79101.

Rassweiler, A. 2010. Mid-Range iPad to Generate Maximum Profits for Apple. iSuppli Estimates. Available at http://www.isuppli.com/Teardowns/News/Pages/Mid-RangeiPadtoGenerate-MaximumProfitsforApple,iSuppliEstimates.aspx (accessed date March 12, 2013).

Ross, S. M. 1983. *Stochastic Processes*. John Wiley & Sons, New York.

Sanchez, R., J. T. Mahoney. 2002. Modularity, flexibility, and knowledge management in product and organization design. *Manag. Mod. Age: Architect. Netw. Organ.* **17**: 549–575.

Schneider, H. 1981. Effect of service-levels on order-points or order-levels in inventory model. *Int. J. Prod. Res.* **19**(6): 615–631.

De Schrijver, S., E. Aghezzaf, H. Vanmaele. 2013. *Aggregate Constrained Inventory Systems with Independent Multi-Product Demand: Control Practices and Theoretical Limitations*. Wiley, Edition 3, New York.

Silver, E., D. Pyke, R. Peterson. 1998. Inventory management and production planning and scheduling. *Int. J. Prod. Res.* **19**(6): 615–631.

Snyder, L. V., Z. M. Shen. 2011. *Fundamentals of Supply Chain Theory*. Wiley Publication, Hoboken, New Jersey.

Sobel, M. J. 2004. Fill rates of single-stage and multistage supply systems. *Manuf. Serv. Oper. Manag.* **5**(1): 41–52.

Song, J. 1998. On the order fill rate in a multi-item. Base-stock inventory system. *Oper. Res.* **45**(6): 831–845.

Subramanian, R., M. E. Ferguson, B. Toktay. 2013. Remanufacturing and the component commonality decision. *Prod. Oper. Manag.* **22**(1): 36–53.

Teunter, R. 2009. Note on the fill rate of single-stage general periodic review inventory systems. *Oper. Res. Lett.* **37**(1): 67–68.

Thomas, D. J. 2005. Measuring item fill-rate performance in a finite horizon. *Manuf. Serv. Oper. Manag.* **7**(1): 74–80.

Thomas, D., D. Warsing. 2007. A periodic inventory model for stocking modular components. *Prod. Oper. Manag.* **16**(3): 343–359.

Turken, N., Y. Tan, A. J. Vakharia, L. Wang, R. Wang, A. Yenipazarli. 2012. The multi-product newsvendor problem: Review, extensions, and directions for future research. Choi, Tsan-Ming, ed. *Handbook of Newsvendor Problems International Series in Operations Research and Management Science*, Volume 176, Springer, New York, 3–39.

Van Mieghem, J. A., N. Rudi. 2002. Newsvendor networks: Inventory management and capacity investment with discretionary activities. *Manuf. Serv. Oper. Manag.* **4**(4): 313–335.

Zhang, J., L. Bai, Y. He. 2007. Fill rate of single-stage general periodic review inventory systems. *Oper. Res. Lett.* **35**(4): 503–509.