Chapter 1
The Multi-product Newsvendor Problem: Review, Extensions, and Directions for Future Research

Nazli Turken, Yinliang Tan, Asoo J. Vakharia, Lan Wang, Ruoxuan Wang, and Arda Yenipazarli

Abstract In this paper, we review the contributions to date for the multi-product newsvendor problem (MPNP). Our focus is on the current literature concerning the mathematical models and the solution methods for the multi-item newsvendor problems with single or multiple constraints, as well as the effects of substitute and complementary products on the stocking decisions and expected profits. We present some extensions to the current work for a stylized setting assuming two products and conclude with directions for future research.

Keywords Multi-product newsvendor • Complementary products • Effects of substitute • Single constraint • Multiple constraints • Future research

1.1 Introduction

The single-item newsvendor problem is one of the classical problems in the literature on inventory management (Arrow et al. 1951; Silver et al. 1998) and the reader interested in a comprehensive review of extant contributions for analyzing the problem is referred to Qin et al. (2011). In this paper, we focus on the multi-product newsvendor problem (MPNP) which can be framed as follows. At the beginning of a single period, a buyer is interested in determining a stocking policy (Q_i) for product i ($i = 1, \ldots, n$) to satisfy total customer demand for each product. For each product i, the customer demand is assumed to be stochastic and characterized by a random variable x_i with the probability density function $f_i(\cdot)$ and cumulative distribution

N. Turken • Y. Tan • A.J. Vakharia (✉) • L. Wang • R. Wang • A. Yenipazarli
Department of Information Systems & Operations Management, University of Florida, Gainesville, FL 32611–7169, USA
e-mail: nazli.z.turken@warrington.ufl.edu; tanyinliang@ufl.edu; asoov@ufl.edu; lan.wang@warrington.ufl.edu; rxwang@ufl.edu; arda.yenipazarli@warrington.ufl.edu

T.-M. Choi (ed.), *Handbook of Newsvendor Problems*, International Series in Operations Research & Management Science 176, DOI 10.1007/978-1-4614-3600-3_1,

function $F_i(\cdot)$. The quantity Q_i is purchased by the buyer for a fixed price per unit v_i. Assuming no capacity restrictions on the purchase quantity and zero purchasing lead time, an order placed by the buyer with the supplier at the beginning of a period is immediately filled. Sales of the product occur during (or at the end of) the single period and for each product i: (a) if $Q_i \geq x_i$, then $Q_i - x_i$ units which are left over at the end of the period are salvaged for a per unit revenue of g_i[1]; and (b) if $Q_i < x_i$, then $x_i - Q_i$ units which represent "lost" sales cost B_i per unit. Assuming a fixed market price of p_i, then the actual end of period profit for the buyer stemming from the sales of each product i is:

$$\pi_i(Q_i, x_i) = \begin{cases} p_i x_i - v_i Q_i + g_i(Q_i - x_i) & \text{if } Q_i \geq x_i \\ p_i Q_i - v_i Q_i - B_i(x_i - Q_i) & \text{if } Q_i < x_i \end{cases}. \tag{1.1}$$

Since the buyer cannot observe the actual end-of-period profit when making his decision at the beginning of the period, the traditional approach to analyze the problem is based on assuming a risk neutral buyer who makes the optimal quantity decision at the beginning of the period by maximizing total expected profits. These profits are:

$$\begin{aligned} E[\pi_i(Q_i)] &= \int_0^{Q_i} [p_i x_i - v_i Q_i + g_i(Q_i - x_i)] f_i(x_i) \mathrm{d}x_i \\ &\quad + \int_{Q_i}^{\infty} [p_i Q_i - v_i Q_i - B_i(x_i - Q_i)] f_i(x_i) \mathrm{d}x_i \\ &= (p_i - g_i)\mu_i - (v_i - g_i)Q_i - (p_i - g_i + B_i) ES_i(Q_i), \end{aligned} \tag{1.2}$$

where $E[\cdot]$ is the expectation operator, μ_i is the mean demand for product i, and $ES_i(Q_i)$ represents the expected units short assuming Q_i units are stocked and can be determined as $\int_{Q_i}^{\infty}(x_i - Q_i) f_i(x_i) \mathrm{d}x_i$. Based on this, the buyer's expected profits for the MPNP are:

$$\begin{aligned} E[\Pi(Q_1, .., Q_n)] &= \sum_{i=1}^{n} E[\pi_i(Q_i)] \\ &= \sum_{i=1}^{n} [(p_i - g_i)\mu_i - (v_i - g_i)Q_i - (p_i - g_i + B_i) ES_i(Q_i)]. \end{aligned} \tag{1.3}$$

Note that (1.1) is separable in each product i. Given that (1.2) is strictly concave in Q, it follows that the first order conditions (FOCs) for optimizing (1.2) are necessary and sufficient to determine the optimal value of Q_i. Based on this, the optimal stocking quantity for each product i (Q_i^*) is set such that:

$$F_i(Q_i^*) = \frac{p_i - v_i + B_i}{p_i - g_i + B_i} \tag{1.4}$$

[1] For obvious reasons, it is generally assumed that $g_i < v_i$.

and the corresponding total profit for the buyer is:

$$E[\Pi(Q_1^*, Q_2^*, .., Q_n^*)] = \sum_{i=1}^{n} \left[(p_i - g_i)\mu_i - (p_i - g_i + B_i) \int_{Q_i^*}^{\infty} x_i f_i(x) \mathrm{d}x_i \right]. \quad (1.5)$$

Our focus in this chapter is on reviewing and extending the current literature related to the MPNP. To start with, Sect. 1.2 reviews the current contributions for analyzing the MPNP with one or more stocking constraints. In Sect. 1.3, we review prior work which focuses on product substitutability in the context of the MPNP. Extensions of the MPNP for complementary/substitute products are described in Sect. 1.4 and finally, in Sect. 1.5, we conclude with directions for future research.

1.2 Buyer Stocking Constraints

1.2.1 Single Constraint

The general problem in this setting is to optimize the total profit in (1.5) subject to the following constraint:

$$\sum_{i=1}^{n} s_i Q_i \leq S, \quad (1.6)$$

where s_i is the storage space or the resource coefficient required per unit of product i and S is the total available storage space or resource. Since (1.3) is strictly and jointly concave in the decision variables, Q_is, and the constraints are linear, one approach to solving this problem would be to start with the solution to the unconstrained MPNP and substitute this solution in the constraint. If the constraint is not violated, then we have an optimal solution. Of course, the issue that needs to be considered is how to solve the problem when the constraint is violated with the solution to the unconstrained MPNP.

Hadley and Whitin (1963) proposed a Lagrange multiplier technique and a dynamic programming solution procedure for finding the optimal order quantity in this setting. The Lagrangian for this context is:

$$L(Q_i, \lambda) = E[\pi(Q_1, ., Q_n)] - \lambda \left(\sum_{i=1}^{n} s_i Q_i - S \right). \quad (1.7)$$

Since (1.7) is strictly and jointly concave in the decision variables, the FOCs are necessary and sufficient to obtain an optimal solution. These FOCs are:

$$\frac{\partial L}{\partial \lambda} = [g_i - v_i]F(Q_i^*) + [1 - F(Q_i^*)][p_i - v_i + B_i] - \lambda s_i, \quad (1.8)$$

$$\frac{\partial L}{\partial Q_i} = \sum_{i=1}^{n} s_i Q_i - S. \tag{1.9}$$

Setting (1.8) equal to 0, the optimal stocking quantity for each product Q_i^* is:

$$Q_i^* = F_i^{-1}\Big(\frac{p_i - v_i - \lambda s_i + B_i}{p_i - g_i + B_i}\Big), \tag{1.10}$$

where $\lambda \geq 0$.

In some practical situations, the optimal Q_i^* will tend to be very small and any attempt to use the above procedure and round the results (to obtain integer values of Q_i^*) could lead to considerable deviations from optimality. To handle this situation, the authors propose a dynamic programming-based procedure but of course, this method is not easily applicable when the number of products (n) is significantly large.

Nahmias and Schmidt (1984) introduced several heuristic methods to solve the MPNP with a single constraint where the lagrange multiplier, λ, is not easy to evaluate. They also included an interest rate, I,which is used in determining the carrying charge per period for the average inventory. Hence, the expected profit including the interest rate can be shown as:

$$E[\pi_i(Q_i)] = (p_i + 0.5Iv_i - g_i)\mu_i - [(1-I)v_i - g_i]Q_i - (p_i + 0.5Ic_i - g_i + B_i)ES_i(Q_i). \tag{1.11}$$

The optimal quantity then becomes:

$$Q_i^* = F^{-1}\left[\frac{p_i - (1+0.5I)v_i + B_i - \lambda s_i}{p_i + 0.5Iv_i - g_i + B_i}\right]. \tag{1.12}$$

Guessing an appropriate value of λ, computing the corresponding values of Q_i and subsequently adjusting the value of λ depending on (1.12) is very time consuming. Thus, four different heuristic methods were introduced. Heuristic 1 finds the solution to the unconstrained problem and adjusts these values until the constraint is satisfied. In heuristic 2, the critical point of the demand distribution is scaled to fit the given volume. Finally, heuristics 3 and 4 are proposed based on the Taylor expansion series of $t_i(\lambda) = \Phi^{-1}(a_i - b_i\lambda)$ and the corresponding λs could be calculated as follows:

$$\lambda_i = \frac{\sum_{i=1}^{n} \mu_i s_i + \sqrt{2\pi}\sum_{i=1}^{n} s_i(a_i - 0.5)\sigma_i - S}{\sqrt{2\pi}\sum_{i=1}^{n} b_i \sigma_i v_i}. \tag{1.13}$$

The procedures listed in this paper are mostly useful for the continuous values of Q_is and are thus appropriate for moderate-to-high demand items.

Lau and Lau (1996) were among the first to observe that using Hadley and Whitin's approach may lead to infeasible(negative) order quantities for some of the considered products when the constraint is tight. They based their work on the classical expected cost minimization problem that was introduced by Hadley and Whitin, where $(v_i - g_i)$ is the unit overage cost and $(p_i - v_i + B_i)$ is the unit underage cost. By rearranging terms of (1.8), we can find the expected net benefit of the marginal unit of product i ($EBMU_i$) at Q_i as:

$$EBMU_i = [p_i - v_i + B_i][1 - F(Q_i^*)] - [v_i - g_i]F(Q_i^*). \tag{1.14}$$

Note that $EBMU/s_i$ is analogous to the λ_i. Lau and Lau introduced a procedure to handle distributions with strictly positive lower bounds as well as distributions with long left tails.

Abdel-Malek et al. (2004) developed the exact solution formulae for uniformly distributed demand and presented a generic iterative method (GIM) when the demand distribution is general. The author considered the total budget as the resource constraint ($\sum_{i=1}^{n} v_i Q_i \leq B_G$). Different from most of the work in the literature, the author assumes there is a leftover cost (disposal fee), where a salvage value is usually considered. In general, if the budget is abundant, the problem could be solved by the unconstrained solution, yet if the budget is tight, we need to apply the Lagrangian-based approach to solve the problem. The value of λ is crucial to solve the problem and the author discusses how to address this under specific and general demand distributions. The formula for λ when the demand is uniformly distributed between a_i and b_i can be written as:

$$\lambda_u = \frac{\sum_{i=1}^{N}(c_i x_i^*) - B_G}{\sum_{i=1}^{N}(b_i - a_i)(v_i^2)/(p_i + h_i)}, \tag{1.15}$$

where h_i is the holding cost. The closed-form expression when the demand is exponential:

$$\lambda_e^{(1)} = \frac{\sum_{i=1}^{N}(c_i x_i^*) - B_G}{\sum_{i=1}^{N}(\mu_i - v_i^2)/(v_i + h_i)}. \tag{1.16}$$

The proposed GIM finds the optimum under uniform distribution and near optimum for other general distributions. GIM first finds the solution without the constraint and checks whether the constraint is satisfied. If the constraint is satisfied, the solution is optimal, if not, a solution that satisfies the constraint is found. Next, the error is estimated, if this is at an acceptable level, the optimal solution is found. As an extension to this paper, Abdel-Malek and Montanari (2005a) also defined the thresholds to help the decision maker in recognizing the tightness of the budget constraint, which can avoid infeasible order quantities by removing products with low marginal utilities. The following equations determine the thresholds depending on the available budget and demand patterns:

Threshold 1:

$$B_G^{(1)} = \sum_{i=1}^{n} v_i F_i^{(-1)}\left(\frac{p_i - v_i + B_i}{p_i - g_i + B_i}\right) = \sum_{i=1}^{n} v_i Q_i^*. \tag{1.17}$$

Threshold 2:

$$B_G^{(2)} = \sum_{i=1}^{n} v_i F_i^{(-1)} \left(\frac{p_i - (\theta^- + 1)v_i + B_i}{p_i - g_i + B_i} \right), \tag{1.18}$$

where $\theta^- = \min_{(i=1,.n)}(\theta_i)$ and notice that θ_i is the marginal utility at the lower limit of the feasible amount of the product to be ordered and could be calculated as follows:

$$\theta_i = \frac{p_i + B_i - (p_i - g_i + B_i)F_i(0)}{v_i} - 1. \tag{1.19}$$

Once the thresholds are defined, the solution procedure for each of the resulting cases can be implemented as shown in the following.

Case 1 $B_G^{(1)} \leq B_G$. In this case, the budget is abundant and the budget constraint is redundant, so we can obtain the optimal solution from the unconstrained problem.

$$F_i(Q_i^*) = \frac{(p_i - v_i + B_i)}{(p_i - g_i + B_i)}. \tag{1.20}$$

Case 2 $B_G^{(2)} \leq B_G < B_G^{(1)}$. In this case, we can relax the nonnegativity constraint and use the Lagrange method to get the optimal solutions.

$$F_i(Q_i^*) = \frac{p_i - (\theta + 1)v_i + B_i}{p_i - g_i + B_i}. \tag{1.21}$$

Case 3 $B_G < B_G^{(2)}$. In this case, as mentioned before, the nonnegativity constraints should be added to the model to avoid the infeasible solution. Furthermore, one or more products will have an order quantity of zero.

To determine the optimal order quantities, one needs to compute the marginal utilities of each product by using (1.19) first and rank them in ascending order. Begin with excluding the product (set order quantity to zero) from the top of the list and continue the exclusion process until the updated budget threshold is less than the previous budget. $B_{(G,i)}$ is the updated budget threshold as well as the lower bound of budget required for including item i in the list, which is expressed by:

$$B_{G,i} = \sum_{i=1}^{n'} v_i F_i^{-1} \left(\frac{p_i - (\theta_i + 1)v_i + B_i}{p_i - g_i + B_i} \right) < B_G. \tag{1.22}$$

n' is the updated number of items on the list. Once this point is reached, the problem becomes tractable again and we can apply the Lagrangian method without nonnegativity constraint to get the optimal solutions.

Several researchers incorporated nonnegativity constraints on the decision variables in their approaches. Erlebacher (2000) developed optimal and heuristic solutions for the classical problem. The first optimal solution refers to the event where each item has a similar cost structure and the demand for each item is from a similar distribution. The second case is when the demand for each item follows a uniform distribution. The first heuristic (H1) is optimal when all of the items have a similar cost structure and similar shaped demand distribution and requires only the mean and variance of each demand distribution and the cost data. The second heuristic (H2) is optimal when the demand is uniformly distributed for each item. The third heuristic (H3) is a modification of (H1) to account for general cost structures based on the form of (H2). The authors use computational experiments to show that (H2) is the most effective one, especially at higher levels of capacity.

Zhang et al. (2009) developed a binary search method to obtain the optimal solution. They defined the marginal benefit function as $r_i(x_i) = (v_i - B_i + \frac{(g_i+B_i)F_i(x_i)}{v_i})$, where $r_i(x_i)$ is a nondecreasing function of x_i, when $x_i \geq 0$ and its inverse is a strictly increasing function of r_i when $1 - \frac{B_i}{v_i} \leq r_i(x_i) < 0$. The authors find that the optimal solution to the constrained problem is the same as the unconstrained optimal solution when the budget constraint is not binding and is less than the unconstrained optimal solution when the budget constraint is binding. If there are nonzero optimal solutions, their marginal benefits should equal each other. When the budget constraint is binding, the optimal solution is $x_i^{(**)}$, and $r^{(**)} = r_i(x_i)$ is the marginal benefit at $x_i^{(**)}$. Zhang and Hua showed that $1 - \frac{B_i}{v_i} \leq r^{(**)} < 0$ and $r^{(**)}$ can be found using a binary search between these values. The algorithm they developed first finds the solution to the unconstrained problem and assesses whether the optimal value leads to a binding budget constraint. If this solution does not satisfy the condition, a binary search procedure is applied. This algorithm can provide an optimal or a near optimal solution to MPNP under any general demand distribution and it can also provide a good approximate solution under discrete demand distributions.

Zhang and Du (2010) studied the MPNP with a capacity constraint, where the products can be outsourced to an external facility at a higher cost. They considered zero-lead time (ZO) and nonzero lead time (NO) strategies. In ZO strategy, the manufacturer makes the decision for the in-house production quantity in the first period, and in the second period, after the demand is realized, the manufacturer outsources the remaining demand with zero lead time. There are no lost sales in this case. In the NO strategy, the manufacturer makes the decision for the in-house production quantity and the outsourcing quantity in the first period. In this strategy, if the demand exceeds the in-house production and outsourcing, there will be backorders or lost sales. The NO strategy assumes that there is no difference in arrival times of the products whether they are outsourced or produced in-house or if a time difference exists, it is assumed that there is no cost to receiving the product earlier than required.

It is assumed that each product has a deterministic production capacity and a random demand. The demand distributions are approximated to exclude the negative values allowing the following assumptions: $F_i(x) = 0$ for all $x < 0$, and $F_i(0) \geq 0$. The expected profit function for the ZO strategy consists of the revenue of product i, salvage value of the excess of product i, less the outsourcing and in-house production cost of product i. This model can be viewed as a parameter-adjusted single-constraint newsvendor model, and can be solved using the methods developed by Zhang et al. (2009). Similarly, the expected profit function of the NO strategy can be written as $\pi_2 = \sum_{i=1}^{n}[(p_i - v_i)Y_{i+}(p_i - d_i)Z_i - (p_i - g_i)\int_0^{Y_i+Z_i} F_i(x_i)\mathrm{d}x_i$, where d_i is the cost of outsourcing one unit, Y_i and Z_i are the decision variables for in-house production and outsourcing, respectively. By analyzing the partial derivatives and the KKT conditions, it is evident that the constrained optimal solution for in-house production will always be smaller than the unconstrained optimal solution for in-house production when there is no outsourcing $(Y_i^* \leq \widetilde{Y})$. The optimal outsourced quantity will also be less than the unconstrained problem solution and the maximum value it can take is $\widetilde{Z} - Y^*$. If the unconstrained optimal in-house production quantity does not exceed the available capacity, everything will be produced in-house. If the unconstrained optimal in-house production quantity exceeds the available capacity, the limited capacity must be fully utilized. Finally, the optimal solutions can be designed in a way that there exists only one product that utilizes both sources of production,and for every other product only one source of production is used. The results are that ZO strategy outperforms the NO strategy when outsourcing costs are equal and managers should try to find a ZO option with low implementing costs to achieve the maximum profit.

Moon and Silver (2000) presented dynamic programming procedures for MPNP, where the budget is represented as the total value of the replenishment quantities. In this paper, the decision variable is the order-up-to level, S_i. There is an inventory level of I_i at the beginning of period i, a fixed ordering cost of A_i, and a variable cost of $G_i^F(S_i^*)$. Initially, it is assumed that there is enough budget to permit each item to be ordered at its optimal. The authors formulate the problem as a minimization of fixed and variable costs $C_i^F(S_i)$, and decide the ordering rule to be: order up to S_i^*, if $I_i < s_i^*$ where $G_i^F(s_i^*) = A_i + G_i^F(S_i^*)$. Moon and Silver, then introduced a restricted budget W and developed a dynamic program to find the optimal order-up-to level. This dynamic program first tries to solve the single period model with a fixed ordering cost for each item separately and defines $\mathbf{P}$ to be the set of items that are profitable to order and reaches to an optimum when the ordering cost is within the budget. This solution method will become time consuming if the number of items or the number of budget constraints are high. Hence, the authors developed two heuristic algorithms. The first one, is a greedy allocation algorithm. At each step, the algorithm reduces the budget until a feasible solution is reached and any remaining budget is filled in a reverse greedy manner. The second algorithm, a two-stage heuristic, tries to assign the budget proportionally to the items in $\mathbf{P}$.

The authors also considered the distribution-free model and assumed that the distribution of the demand belongs to the class $\mathbf{F}$ cumulative distribution functions with mean μ_j and variance σ_j^2. This approach requires finding the most unfavorable distribution in $\mathbf{F}$ for each S. Then, the objective function becomes $\min_{(S_1,\ldots,S_m)} \max_{(F\in F)} \sum_{i=1}^{m} C_i^F(S_i)$. The authors rewrite the cost function as $\sum_{i=1}^{m} G_i^W(S_i) + A_{(i)} 1_{(S_i > I_i)}$ using the proposition from Gallego and Moon (1993) indicating that a distribution satisfying $E[D_i - S_i]^+ \leq \frac{1}{2}\{\sqrt{[(\sigma_i^2) + (S_i - \mu_i^2)^2]} - (S_i - \mu_i)\}$ can always be found. In this cost function,

W denotes a worst case distribution function of the demand. The optimal solution can be found through backward recursive equations. The use of this distribution-free solution is justified when the expected value of additional information $(EVAI) = \sum_{i=1}^{m} C_i^N(S_i^W) - \sum_{i=1}^{m} C_i^N(S_i^N)$ is low. They mentioned two heuristics can also be modified to solve the distribution-free approach; however, this has not been studied in this paper.

Shao and Ji (2006) studied the multi-item newsvendor problem where the demand is fuzzy. They defined the profit for product i to be:

$$f(x_i, x_i(\xi_i)) = \begin{cases} (p_i - v_i)Q_i & \text{if } Q_i \leq x_i \\ (p_i - g_i)x_i - (v_i - g_i)Q_i) & \text{if } Q_i \geq x_i \end{cases} \tag{1.23}$$

and the total profit as $F(x, X(\xi)) = \sum_i^n f(x_i, x_i(\xi_i))$ subjected to a budget constraint. They adopted the credibility theory and defined the credibility of a fuzzy event as $Cr(x_i \geq p) = \frac{1}{2}[Pos(x_i \geq Q_i) + Nec(x_i \geq Q_i)]$, where $Pos(x_i \geq Q_i) = sup_{u \geq Q_i} \mu(u)$ and $Nec(x_i \geq Q_i) = 1 - sup_{u<Q_i} \mu(u)$. The maximum expected profit of the newsvendor problem (MEP) is $E[F(Q^*, x_i)]$ when $E[F(Q^*, x_i)] \geq E[F(Q, x_i)]$ holds for all feasible Q. In the cases where a confidence level, α, is set as a safety margin, α-maximum profit is $\overline{F}$, where $\max(\overline{F}|Cr(F(Q^*, x_i) \geq \overline{F}) \geq \alpha) \geq \max(\overline{F}|Cr((Q, x_i) \geq \overline{F} \geq \alpha)$. The most maximum profit (MMP) is $F(Q^*, x_i)$, when $Cr((Q^*, x_i) \geq F_0) \geq Cr((Q, x_i) \geq F_0)$ where F_0 is the predetermined profit. The authors formulate three models to represent the problem. The first one, the expected value model, maximizes the expected value operator of the fuzzy event with nonnegativity and the budget constraints. The second one, chance constraint model, maximizes $\alpha - MP$ subject to credibility, nonnegativity, and budget constraints. The third model, dependent chance programming, maximizes the credibility not less than the predetermined profit with budget and the nonnegativity constraints. In this paper, a hybrid intelligent algorithm combining fuzzy simulation and genetic algorithm is introduced and numerical examples are provided to display the performance of this algorithm with the three different models mentioned.

Lau and Lau (1988) studied an MPNP, where the objective is to maximize the probability of a given target profit. They assumed that the shortage cost is zero and also showed that any problem with $g_i \geq 0$ can be converted to another one without salvage value. The objective is to maximize $P_T = Prob(\text{Total Profit} \geq \text{Target Profit(T)})$. They consider three different approaches to find the optimum; use simulation to find Q_i^*s and repeat to find the maximizing P_T for different pairs of Q_i^*s, derive an expression for P_T and use a "hill-climbing" method to find Q_i^*s or analytically solve for the FOCs of P_T. They execute approach 2 and 3, and perform some numerical studies for specific demand distributions. In the first case, they define T_m as $(p_1 - v_1)\mu_1 + (p_2 - v_2)\mu_2$, and the target profit as T,which is set to be T_m, $0.5T_m$, or $0.25T_m$. The products are assumed to have identical parameters and demand distributions. Lau proved previously in a single product model that

$$Q_i^* = T/(p - v). \tag{1.24}$$

In order to find the optimum, they first find the Q_I^* that satisfy (1.24) for the single item case. The results show that when $Q_I = Q_i$ the individual and global optimal values are the same. Otherwise, if $P_T^* < P_I$, the optimal for individual products do not give the global optimum. If $P_T^* > P_I$ and $Q_I \neq Q_i$, a reward policy can be implemented to drive the subordinates to achieve the maximum global probability. The authors applied this procedure for normally distributed demands as well. While deriving the mathematical expression for P_T in approach 3, they introduced two different situations. Situation B happens when $p_iQ_i \geq T + v_1Q_1 + v_2Q_2$ for both products and Situation A happens if $p_iQ_i \geq T + v_1Q_1 + v_2Q_2$ holds for one product, and $p_iQ_i < T + v_1Q_1 + v_2Q_2$ holds for the other.

- Situation A

  Range 1: $0 \leq x_1 \leq L_1$ where $L_1 = (T + v_1Q_1 + v_2Q_2 - p_2Q_2)/p_1$. We know that the profit from product 2 is $p_2Q2 - v_2Q_2$; thus, product 1 must contribute $T - (p_2Q2 - v_2Q_2)$. If the demand is in this range, $P_{T1} = 0$.
  Range 2: $L_1 \leq x_1 \leq Q_1$. In this range, the profit from product 1 is $p_1x_1 - v_1Q_1$ and the profit from product 2 is $T - (p_1x_1 - v_1Q_1)$. Hence, the probability of achieving T when the demand is in range 2 is: $P_{T2} = \int_{L_1}^{Q_1} f_1(x_1)[1 - F_2(\frac{T+v_1Q_1+v_2Q_2-p_1x_1}{p_2})]\mathrm{d}x_1$.
  Range 3: $Q_1 \leq x_1 \leq \infty$. In this range, the profit from product 1 is constant, $(p_1 - v_1)Q_1$, and the profit from product 2 is $T - (p_1 - v_1)Q_1$. Thus, the probability of achieving T when the demand is in range 3 is: $P_{T3} = \int_{Q_1}^{\infty} f_1(x_1)[1 - F_2(\frac{T+v_1Q_1+v_2Q_2-p_1Q_1}{p_2})]\mathrm{d}x_1$.

- Situation B

  Range 1: $0 \leq x_1 \leq L_2$ where $L_2 = (T + v_1Q_1 + v_2Q_2)p_1$. Hence,
  $P_{T1} = \int_0^{L_2} f_1(x_1)[1 - F_2(\frac{T+v_1Q_1+v_2Q_2-p_1x_1}{p_2})]\mathrm{d}x_2$.
  Range 2: $L_1 \leq x_1 \leq \infty$. In this range, the probability of achieving T is:
  $P_{T2} = 1 - F_1(L_2)$.

The authors then derive the optimal ordering quantities assuming that the parameters for both items are equal and the demand follows a uniform distribution. Finally, they consider the case where the selling prices of each item is different. They found using approach 2 that if $p_1 < p_2$, then $Q_1^* > Q_2^*$ when T is small and $Q_1^* < Q_2^*$ when $T \geq T_m$.

Vairaktarakis (2000) mentioned in his paper, "…..along with the traditional risk averse attitude, the managers render minimax regret approaches very important in identifying robust solutions, i.e., solutions that perform well for any realization of the uncertain demand parameters." Based on this, he presents a number of minimax regret formulations for the multi-item newsvendor problem with a single budget constraint, when the demand distribution is completely unknown. Demand uncertainty is captured by means of discrete and continuous scenarios.

In discrete demand scenarios, let $D^S(i)$ be the collection of all possible demand realizations for item i, $i = 1, 2, \ldots, n$. Then, the solution to any of the multi-item problems that will be stated below must be n-tuple in $D^S(1) \times D^S(2) \times \ldots \times D^S(n)$. We consider three different objective functions.

- *Absolute robustness*. This approach attempts to find an n-tuple of order quantities that maximize the worst case profit over all possible demand realizations.

$$\max_{(Q_1 \ldots Q_n) \in D^S(1) \times D^S(2) \times \ldots \times D^S(n)} \; \min_{(d_1 \ldots d_n) \in D^S(1) \times D^S(2) \times \ldots \times D^S(n)} \sum_{i=1}^{n} \pi_i(Q_i, d_i)$$

$$s.t. \quad \sum_{i=1}^{n} c_i Q_i \leq W.$$

- *Robust deviation*. This formulation provides a solution that minimizes the maximum profit loss due to demand uncertainty. The objective function is:

$$\min_{(Q_1 \ldots Q_n) \in D^S(1) \times D^S(2) \times \ldots \times D^S(n)} \; \max_{(d_1 \ldots d_n) \in D^S(1) \times D^S(2) \times \ldots \times D^S(n)}$$
$$\times \sum_{i=1}^{n} \pi_i(d_i, d_i) - \pi_i(Q_i, d_i) \quad s.t. \quad \sum_{i=1}^{n} c_i Q_i \leq W,$$

  where $\pi_i(d_i, d_i) - \pi_i(Q_i, d_i)$ stands for the profit that could be realized if there was no demand uncertainty less the profit made for the order quantity Q_i.
- *Relative robustness*. This minimizes the relative profit loss per unit of profit that could be made if there was no demand uncertainty.

$$\min_{(Q_1 \ldots Q_n) \in D^S(1) \times D^S(2) \times \ldots \times D^S(n)} \; \max_{(d_1 \ldots d_n) \in D^S(1) \times D^S(2) \times \ldots \times D^S(n)}$$
$$\times \sum_{i=1}^{n} \frac{\pi_i(d_i, d_i) - \pi_i(Q_i, d_i)}{\pi_i(d_i, d_i)} \quad s.t. \quad \sum_{i=1}^{n} c_i Q_i \leq W.$$

In the continuous demand scenario, the demand in (1.4) is bounded by $\underline{D}$ and $\overline{D}$. Then the minmax problem becomes:

$$\max_{Q_i} \; \min_{d_i \in [\underline{D_i}, \overline{D_i}]} \sum_{i=1}^{n} \pi_i(Q_i, d_i)$$
$$s.t. \quad \sum_{i=1}^{n} c_i Q_i \leq W,$$
$$Q_i \in [\underline{D_i}, \overline{D_i}].$$

This problem can be reduced to a continuous knapsack problem, and solved by the proposed algorithm in this paper. Then, the optimal quantity is:

$$Q_* = \frac{(v_i - g_i)\underline{D} + (p_i - v_i + B_i)\overline{D}}{p_i - g_i + B_i}. \tag{1.25}$$

Similarly, Choi et al. (2011) considered a risk-averse MPNP under the law-invariant coherent measures of risk. They have shown that for heterogeneous products with independent demands, increased risk aversion leads to decreased orders, and derived closed-form approximations for the optimal order quantities. Also, they have shown that risk-neutral (maximize the expected profit) solutions are asymptotically optimal under risk aversion as the number of products tends to be infinity. This result has an important business implication: companies with many products or product families with low demand dependence need to look only at risk-neutral solutions even if they are risk averse. For a risk-averse newsvendor with dependent demands, they showed that in a two-product model with positively dependent demands, the optimal order quantities are lower than for independent demands, while for negatively dependent demands, the optimal order quantities are higher.

In another paper where risk was taken into consideration, Ozler et al. (2009) consider a single-period MPNP, where a retailer determines the optimal order quantities of N different products having stochastic demand. Furthermore, they integrate risk considerations (i.e., the risk of earning less than a desired target profit or losing more than an acceptable level due to demand uncertainty) through a Value at Risk (VaR) approach. VaR is a measure of downside risk and is defined as the probability of earning lower than the target profit value is less than or equal to a threshold probability value.

In order to illustrate the approach, the authors first derive a compact expression for the distribution of the profit for two products with a joint demand distribution, and explicitly derive the VaR constraint in terms of the decision variables Q_1 and Q_2. The formulated problem turns out to be a mixed-integer programming formulation with a nonlinear objective function under mixed linear and nonlinear constraints. They analyze the conditions for the feasibility of this problem and present a mathematical programming formulation that determines the optimal order quantities. The authors also consider a correlated demand structure, and by solving the two-product problem, they show that the expected profit is higher when two products with negatively correlated demands are used under a VaR constraint. On the other hand, when the VaR constraint is ignored, demand correlations have no impact on the expected profit.

The authors also attempt to extend the procedure outlined for the two product case to more than two products. In this line, they develop an approximation method in case where there are N products with independent, normally distributed demands. They utilize the central limit theorem to determine the distribution of profit approximately and express VaR constraint by using the normal approximation. Similar to the two-product setting, they analyze the feasibility conditions and present a mathematical programming approach that yields optimal order quantities. The case of the MPNP with a correlated demand structure is left for future research.

Mieghem and Rudi (2002) introduce a class of models called *newsvendor networks*, which allow for multiple products, multiple processing, and storage points and investigate how their single-period properties extend to dynamic settings. Such a model provides a parsimonious framework to study various problems of stochastic capacity investment and inventory management, including assembly, commonality, distribution, flexibility, substitution, and transshipment.

Consistent with the previous multidimensional newsvendor models, the newsvendor networks are defined by a linear production technology, which describes how inputs (supply) is transformed into outputs of fill end–product demand, a linear financial structure, and a probability distribution of end–product demand. This paper continues by incorporating multiple storage points into the multidimensional newsvendor model.

We describe the features of a newsvendor network briefly. Before the demand is realized, a set of "ex-ante" activities are performed on the inputs and their results are stored in "stocks" or inventories. After the demand is realized, "ex-post"activities process stocked inputs into demanded outputs using resources. In addition to being constrained by demand, the sales or the output rate is also constrained both by input stock levels and by the resource capacities, denoted by vectors S and K. The ex-ante activities generate the cost vector, v; the ex-post activities generate the marginal value vector $p-v$; the units carried over to subsequent period incur a holding cost h. Let c_K denote per unit capacity investment cost and x denote the flow units. For example, activities 3 and 2 deplete stocks 1 and 2, respectively, and consume Resource 2's capacity at a rate of α^{-1} and 1; activity 1 depletes stock 1 and consumes resource 1. Hence, the inventory constraints are: $x_1 + x_3 \leq S_1$ and $x_2 \leq S_2$, while the capacity constraints are $x_1 \leq K_1$ and $x_2 + \alpha^{-1}x_3 \leq K_2$. Newsvendor networks are thus about three decisions: capacity investment decisions K, input inventory procurement decisions S, and activity decisions $x(K,S,D)$.

The objective is to maximize the expected operating profit, which is the net value from processing minus the shortage penalty cost and holding cost:

$$\Pi(K,S) = E \max_{x \in X(K,S,D)} \left[(p-v)X - B(D-R_D x)^+ - h(S-R_S x)^+\right],$$

where R_S and R_D are input–output matrices, and A is the capacity consumption matrix. The set of feasible activities are constrained by supply S, demand D, and capacity K:

$$X(K,S,D) = x \leq 0 : R_S x \leq S, R_D x \leq D, Ax \leq K.$$

The expected firm value to be maximized is :

$$V(K,S) = \Pi(K,S) - vS - c_K K.$$

This paper presented single period optimality conditions and showed that they retain their optimality in a dynamic setting, so that a stationary base-stock policy is optimal. Besides, it also shows that as in most inventory settings, lost sales are more tractable in newsvendor networks than backlogging. The discussion suggests that the culprits are discretionary activities or joint ex-post capacity constraints, both of which make the order-up-to levels of inputs dependent on backlog in a nonlinear manner.

1.2.2 Multiple Constraints

Similar to the MPNP with a single constraint, the MPNP with multiple constraints has also been investigated by a few researchers. Ben-Daya and Raouf (1993) first presented an analytical solution procedure for a two-constraint multi-item newsvendor problem in which all items' demands are uniformly distributed. Lau and Lau (1995) presented a Lagrangian-based numerical solution procedure of a multi-item newsvendor problem with multiple constraints. Their proposed solution procedure is an adaptation of the "Active Set Methods" which consists of two basic components:

- *Component A.* For a given subset W(called the "working set") of all the resource constraints, component A solves the equality-constrained problem:

$$\begin{aligned} &\text{Max} \sum_{i=1}^{N} E[\pi_i(Qi)] \\ &\text{s.t.} \sum_{i=1}^{N} r_{i,j} Q_i \leq R_j, j = 1, 2, \ldots M, \end{aligned} \tag{1.26}$$

  where $r_{i,j}$ is the coefficient of resource j of item i and $R_j$is the amount available of resource j.
- *Component B.* This is the procedure for defining and updating the working set W for each altered component A. This component A and component B cycle is repeated until the optimal condition is met. The authors provide mathematical details and numerical examples to validate this method.

Lau and Lau (1997), in the sequel of their earlier works, proposed a three-step procedure that used subjective probability elicitation to supplement whatever empirical data is available to construct demand distribution functions. Since the typical multi-item newsvendor problem solution procedure requires many repeated evaluations of the demand's inverse cdfs', the authors suggest using Tocher's general "inverse cdf" to fit the distribution function:

$$\begin{aligned} F_T^{-1}(P) &= D \\ &= a + bp + cp^2 + \alpha(1-p)^2 ln(p) + \beta p^2 ln(1-p), \end{aligned} \tag{1.27}$$

where five parameters (a,b,c,α, and β) could be determined by least-squares fitting. Similar to the normal distribution, $F_T^{-1}(P)$ has a negative tail, which is eliminated by using the following modification:

$$F_M^{-1} = D = \max[0, F_T^{-1}(P)]. \tag{1.28}$$

Abdel-Malek and Areeratchakul (2007) and Areeratchakul and Abdel-Malek (2006) developed an approximate solution procedure to deal with this type of problem. It is based on a triangular presentation of the areas resulting from integrals that are included in the objective function, which facilitates expressing the objective function in quadratic terms. One can solve this problem using familiar linear programming packages. The authors have shown that the objective function can be expressed in the following quadratic form:

$$\operatorname{Max} Z = \sum_{i=1}^{N} \left(A_i^{(.)} x_i^2 + B_i^{(.)} x_i + C_i^{(.)} \right), \tag{1.29}$$

where expressions $A_i^{(.)}$, $B_i^{(.)}$, and $C_i^{(.)}$ are constants to be determined for each product according to its demand probability distribution. In order to get the quadratic form above, we first need to approximate the integral of the cumulative distribution function using triangular approach as:

$$\int_0^{x_i} F(D_i)\mathrm{d}D_i \approx \frac{1}{2}(x_i - x_{l,i})(\Delta_i(x_i - x_{l,i})), \tag{1.30}$$

where $x_i - x_{l,i}$ is the length of the triangle base, $F(x_i) = \Delta_i(x_i - x_{l,i})$ is the height of the triangle with respect to x_i, and Δ_i represents the slope of the triangle. For more details about these parameters under different distribution functions, readers can refer to the paper.

Abdel-Malek and Montanari (2005b) discussed a solution procedure for the MPNP with two constraints. The methodology in the paper is based on analyzing the dual of the solution space as defined by the constraints of the problem. In order to avoid infeasible (negative) solutions, the authors propose that we begin by defining the possible regions of the dual of the solution space. The corresponding solution method is selected based on the area which the resource point is in. Finally, the authors present two numerical examples to illustrate the application of the proposed approach; the first one considers the case where only one of the constraints is binding, and the second one analyzes the case where both constraints are binding.

In addition to the methods mentioned above, Niederhoff (2007) utilized the separability of the objective function and used convex separable programming to minimize the expected cost and calculate the optimal order quantities. Due to the properties of the piecewise linear approximation method, this problem can be studied without any specific distribution. This method also provides sensitivity analysis which can give us some important insights.

1.2.3 Other Constrained MPNP Approaches

In addition to the classical constrained approaches, several authors focused on the applications of the MPNP to address specific issues. Khouja and Mehrez (1996) formulated a MPNP under a storage or budget constraint such that progressive multiple discounts are offered to sell excess inventory. They provided different algorithms depending on whether the optimal order quantities are large or small. The authors assumed that there is a perfect and positive correlation between the demand at the jth discount price and the demand at the nondiscount price. If the demand for the product at the nondiscount price is high, then discounting the price of the product results in a proportionally high additional demand. Observations on the solution to the constrained problem show that storage (or budget) constraint in an MPNP reduces the service levels (i.e., probability of satisfying demand) and order quantities of all products, when compared to the corresponding levels for the unconstrained problem. Furthermore, the numerical examples that compare multiple and single discount solutions indicate that using multiple discounts instead of discounting just once to the salvage value may result in a different optimal solution.

Shi and Zhang (2010), Shi et al. (2011) and Zhang (2010) investigated the MPNP with supplier quantity discounts and a budget constraint, and the effect of these two features on the optimal order quantities. In this line, Zhang presented a mixed integer nonlinear programming model to formulate the problem. The proposed Lagrangian relaxation approach is demonstrated by means of numerical tests. Finally, the problem is extended to multiple constraints, including space or other resource limitations. It is assumed that suppliers provide all-quantity discounts, and the newsvendor faces uncertain demand for multiple products. Besides, the probability density function for each product is assumed to be given.

To solve the problem, the authors use the Lagrangian heuristic and present methods to find upper and lower bounds, as well as an initial feasible solution. They relax the budget constraint (instead of discount constraints that potentially give a tighter dual bound) as it results in a classical newsvendor subproblem with discount constraints. The computational results indicate that the algorithm is extremely effective for the newsvendor model with supplier quantity discounts and a budget constraint (in terms of both solution quality and computing time). The computational results for the multi-constraint case also indicate that the proposed approach performs well for the problems with multiple constraints.

In a different extension, Chen and Chen (2010) developed a multi-product newsvendor model under a budget constraint with the addition of a reservation policy. Reservation policies reduce the demand uncertainty of newsvendor-type products. Under the reservation policy studied in this paper, a discount rate is offered to consumers in order to induce them to make a reservation and buy in advance. The authors propose a general algorithm, namely the MCR algorithm, which finds the optimal order quantity and the discount rate necessary to maximize the total expected profit under the budget constraint. In order to illustrate the efficiency of the proposed algorithm, MCR, they solve a numerical example and compare the classical multi-product budget-constraint newsvendor model (CMC model) with the multi-product budget-constraint newsvendor model with the reservation policy. Numerical results show that the total expected profit obtained from the MCR is greater than that of CMC. This is tied to the reservation policy proposed in the model. The difference between the profits of these two models is treated as the value of information. Thus, we can conclude that the decision to adopt the reservation policy depends on the trade-off between the information value and the cost incurred to establish the willingness function and extra-demand functions.

Aviv and Federgruen (2001) address the multi-product inventory system problem with random and seasonally fluctuating demands. Moreover, they extend the analysis to a multi-echelon problem, two stages specifically. This paper contributes to the literature on *delayed product differentiation* strategies and makes an assumption that "Demands in each period follow a given multivariate distribution with arbitrary correlations between items." In addition, "....as in virtually all inventory models, the demands in different periods are independent and their distributions are perfectly known." Unsatisfied demand is backlogged; each cost component of a product is a function of the product's inventory position(including inventory on hand, blanks being transformed into units of the final product and minus the backlogs). The objective is to minimize expected discounted cost over a finite or infinite horizon or to minimize the long-run average value.

To include all the cost components, this paper defines a expected value of costs for jth item in a period of type k as

$$\overline{G}_j^k = \alpha^j E h_j \left(\left[y_j - d_j^k - d_j^{k+1} \ldots - d_j^{k+l_j} \right]^+ \right)$$
$$+ p_j \left(\left[d_j^k + d_j^{k+1} + \ldots + d_j^{k+l_j} - y_j \right] + \right),$$

where y_j is the inventory position of item j at the beginning of a period, d_j^k is the demand at period k for item j, h_j is the holding cost, and α^j is the discount factor. The model can be formulated as a Markov Decision Process with countable state space $S = \{(x,k)$, x is integer, $k = 1,...,K\}$ and finite action sets $A(x,k) = \{y : x \leq y$ and $\sum_{j=1}^{l} y_j \leq \sum_{j=1}^{l} x_j + b^k\}$. To solve this problem, the authors propose a lower-bond approximation and heuristic strategies. In the case of a 2-stage echelon, i.e., production has positive lead time, they simply modified $R^k(.)$, which is the one-step cost function in a single-item model, and introduced the system-wide echelon inventory position of blanks.

Chung et al. (2008) considered the items with short life cycles or seasonal demands. They developed a two-stage, multi-item model incorporating the reactive production that employs a firm's internal capacity. Reactive capacities are preallocated to each item in preseason stage and cannot be changed during the reactive stage. The objective is expected profit maximization. A simple algorithm for computing optimal policies is presented. This paper aims to help managers understand how employing internal capacity during reactive stage can reduce the impact of the poor demand forecasts. Without fixed costs, the optimal production vector for the reactive stage is a simple function of the production vector, Q, for the preseason stage and the capacity allocation vector, Z, for the reactive stage. By analyzing the KKT conditions, the optimal solution and the Lagrange multiplier, λ^*, can be determined.

Casimir (2002) used MPNP to determine the value of incomplete information. He focused on the value of information in three newsvendor models: the basic model with no constraints, the model with budget or capacity constraints, and the model with substitutability. The value of incomplete information is considered in the form of product-mix information and global information. Product-mix information implies total demand is unknown, but the distribution over products is known exactly. In this case, the overall optimal order quantity is determined initially, and then the optimal order quantity for each single item is determined from the actual value. In the case of global information, total demand is known, but its distribution over the products is unknown. Then, the optimal order quantity for each individual item with the given total demand has to be determined. Here, the authors compute the value of incomplete information by comparing the expected profits of the two cases, and do not consider the performance criteria. Besides, rather than the computation of optimal order quantities, results are computed numerically to provide a research framework for the value of information. Their assumptions are: (1) demand is normally distributed, (2) demand for different items is independent, (3) the salvage value for unsold items is zero, (4) there is no penalty for unmet demand, (5) price, cost, average demand, and standard deviation of demand for all products are the same, (6) for the model with budget constraint, only two items are considered, (7) analyzing substitutability, only a two-item newsvendor problem is considered, and it is assumed that the customer takes a single unit of the substitution product, and (8) substitutability is assumed to be symmetric.

In the model without additional complications, it is shown that the value of product-mix information increases with the number of items, whereas the value of global information decreases with the number of items. The value of both product-mix information and global information decreases with a budget constraint. Furthermore, the value of perfect information also decreases with a budget constraint. The probability of substitution decreases the value of product-mix information such that it is zero with complete substitution, and increases the value of global information so that it is equal to the value of perfect information with complete substitution.

Finally, Zhang and Hua (2010) consider a system where the products are procured from the supplier with a fixed-price contract. Under this procurement strategy, the retailer does not order enough products to avoid the risk raised from demand uncertainty (i.e., lower realized demand). The authors here apply a portfolio approach to MPNP under a budget constraint, where the retailer's procurement strategy is designed as a portfolio contract. In this case, each newsvendor product can be procured from the supplier with dual contracts: a fixed-price contract and an option contract. The retailer can lower the inventory risk by utilizing the flexibility of the option contract. On the other hand, it in turn results in additional costs compared to fixed-price contracts, since unit reservation and execution cost of option contract is typically higher than unit cost of a fixed-price contract. In the paper, the objective is to maximize the total expected profit of the retailer through determining the optimal order quantities of products procured with portfolio contracts. The authors consider a single-period model and assume that the retailer sells n products with independent and stochastic demands. All demands are considered to be nonnegative. The portfolio contract consists of a fixed-price contract and an option contract. In the fixed-price contract, the retailer pays a unit fixed cost for each product he procured from the supplier. In the option contract, to reserve certain order quantity, the retailer prepays a unit reservation cost up-front. Then, the retailer pays an execution cost for each unit purchased up to the option reservation level. The retailer loses the initial payment if he does not exercise the option. Related to those, it is further assumed that:

- The total cost of option contract (reservation plus execution cost) is larger than the cost of fixed-price contract.
- The reservation cost of option contract is smaller than the pure procurement cost of the fixed-price contract.

Following the problem formulation, the authors establish the structural properties of the optimal solution (e.g., the concavity of expected profit function) and propose a polynomial solution algorithm of o(n) order. The main advantage of the proposed algorithm is that it does not depend on a specific demand distribution and it is applicable to general continuous demand distributions. Finally, they conduct numerical studies and sensitivity analysis to show the efficiency of the proposed algorithm, as well as compare three procurement contracts: fixed-price contract (FC), option contract (OC), and portfolio contract (PC). It is evident that the newsvendor model with PC, generates significant improvement compared to FC and OC models. Furthermore, it is shown that following the increase in the available budget, the performance gap between FC and PC models decreases, while the gap between OC and PC increases.

Vaagen and Wallace (2008) study risk hedging in fashion supply chains. They consider two states of the world: $State1$ when a variant of a product becomes popular and the others go out of fashion, and $State2$ is when the reverse happens. In this paper, Vaagen and Wallace provide a portfolio building decision model under uncertainty by combining The Markowitz and the newsboy models into a stochastic optimization model. This model tries to minimize the profit risk using semi-variance. The results of the different scenarios show that hedging portfolios gives any company a competitive advantage. We can also conclude that the uncertain information such as demand estimates and trend information for a certain group of products are not as important in the fashion industry as it is in other industries. The best approach in this case is to define and release hedge portfolios. This model can be extended to include substitutability, which is discussed by Cheng and Choi (2010).

1.3 Substitute Products

Retailers often offer product substitutes to prevent customer loss. This substitution can be perfect, partial, or downward. Most of the early works used two-way substitution and introduced heuristics to find the optimal order quantities. Recent works, however, focus more on one-way substitution. This type of substitution arises in real life, for example, in the semiconductor industry; a higher capacity chip can be used to satisfy demands for lower capacity chips. Current literature in this stream can be classified as those focusing on one-way substitution or two-way substitution.

1.3.1 One-Way Substitution

Bassok et al. (1999) concentrated on full downward substitution among the various structures of substitution. Considering that there are N products and N demand classes, full downward substitution implies that excess demand for class i can be satisfied using stocks of product j for $i \geq j$. The authors discuss a two-stage profit maximization formulation for the multi-product substitution problem. In the first stage, the orders are placed (before demands are realized) , and in the second stage the products are allocated to demands (after demand is observed) (i.e., allocation problem). The authors assume that there are N products and N demand classes, and the demands for each class are stochastic. The order, holding, penalty, and salvage costs are assumed to be proportional, and the revenue is linear in the quantity sold. It is further assumed that the substitution cost is proportional to the quantity substituted. Delivery lags and capacity constraints are ruled out. Finally, it is assumed that the revenue earned for each unit of satisfied demand in class i depends only on i and not on the type of product j used to satisfy the excess demand. The authors assume that: (a) it is more profitable to satisfy unmet demand of class i than of class j, for $i < j$; (b) the effective salvage value of product i is not less than that of product j, for $i < j$; and (c) the substitution of product i for demand class j is profitable.

Let $I(\overrightarrow{x})$ be the maximum single period profits and $P(\overrightarrow{x},\overrightarrow{y})$ be the expected single period profits when the starting inventory before placing the order is $\overrightarrow{x}$ and after ordering is raised to $\overrightarrow{y}$. Then, $I(\overrightarrow{x}) = \max_{(}\ \overrightarrow{y} \geq \overrightarrow{x})P(\overrightarrow{x},\overrightarrow{y})$. Let $\overrightarrow{d} = (d_1,\ldots,d_N)$ be a vector of realized demands. Define $F(\overrightarrow{d}) = F_{1,2,\ldots,N}(d_1,\ldots,d_N)$ as the joint distribution of demands from class 1 to N. Let $G(\overrightarrow{y},\overrightarrow{d})$ be the profits for a given stock level, $\overrightarrow{y}$, and the realized demand, $\overrightarrow{d}$. Let $w_i j$ be the quantity of product j allocated to the demand class j. Then

$$P(\overrightarrow{x},\overrightarrow{y}) = -\sum_{k=1}^{N} c_k(y_k - x_k) + \int_{R_+^N} G(\overrightarrow{y},\overrightarrow{d})\mathrm{d}F(\overrightarrow{d}), \tag{1.31}$$

where:

$$G(\overrightarrow{y},\overrightarrow{d}) = \max_{u_i,v_i,w_{ji}} \sum_{i=1}^{N}\sum_{j=1}^{i} a_{ji}w_{ji} + \sum_{i=1}^{N} s_i v_i - \sum_{i=1}^{N} \pi_i u_i.$$

Subject to

$$u_i + \sum_{j=1}^{i} w_{ji} = d_i \text{ for } i = 1,\ldots,N, v_i + \sum_{i=1}^{N} w_{ji} = y_i \text{ for } i = 1,\ldots,N. \tag{1.32}$$

The authors present a greedy algorithm for the allocation problem, and give a new and compact notation of writing the first differentials of the profit function with respect to stock levels. They prove that given a starting inventory level, the allocation algorithm will maximize profits in $G(\overrightarrow{y}, \overrightarrow{d})$. In addition, the profit function $P(\overrightarrow{x}, \overrightarrow{y})$ is proven to be concave and submodular. They also propose an iterative algorithm to compute the order points for a two-product problem, and develop bounds on the optimal order points. Finally, they present a computational study for the two-product problem and show that the benefits of solving for the optimal quantities, when substitution is considered at the ordering stage, are higher with high demand variability, low substitution cost, low profit margins, high salvage values, and similarity of products in terms of prices and costs.

Smith and Agrawal (2000) have analyzed the impact of retail assortments on inventory management and customer service. They focused on product variety in retailing environment, where the customers can often be satisfied by one of several items, e.g., light colors of T-shirts in apparel. In this paper, they develop a probabilistic demand model for items in an assortment that capture the effects of substitution and provide a methodology for selecting item inventory levels so as to maximize total expected profit, subject to given resource constraints. Because of substitution, the inventory levels for products in an assortment must be optimized jointly.

They consider inventory policies that reinitialize at the start of each fixed cycle, assuming lost sales occur if there is a stockout before the end of the cycle. The authors also analyze several illustrative numerical examples to demonstrate the insights, such as, substitution effects can reduce the optimal assortment size, and policies of ignoring substitution effects can be less profitable than those that explicitly incorporate substitution effects. Similarly, Shah and Avittathur (2007) studied the effects of retail assortments on inventory control. Different from Smith,he defined a demand cannibalization and substitution index and assumed the demand to be a Poisson process (similar to Anupindi et al. 1998). The numerical results showed that when the fixed cost and salvage value of a customized product is high and its incremental profits are low, it is not feasible to carry customized products.

In addition, Smith and Agrawal (2000) also studied a “static” substitution model. They assumed that the choice by the customer is independent of the current inventory levels and the customer does not accept a second choice. Mahajan and van Ryzin (2001) used a choice process based on a utility that is assigned by the customer to each product. This utility is interpreted as the net benefit to the customer from purchasing or not purchasing a product. In this case, the information available to the retailer is only the probability of a sample path $\omega = \{U_t : t + 1, \ldots, T\}$, where T is the number of customers. The number of sales is dependent on the initial inventory level and the sample path. The authors then introduce a sample path gradient algorithm to obtain the optimal results.

Dutta and Chakraborty (2010) studied the newsboy problem with one-way substitution where the demand is fuzzy. The membership function of demand of product i is represented as:

$$\mu_{\widetilde{D_i}(x)} = \begin{cases} L_i(x) = \frac{x-\underline{D_i}}{D_i-\underline{D_i}}, & \underline{D_i} \leq x \leq D_i \\ R_i(x) = \frac{\overline{D_i}-x}{\overline{D_i}-D_i}, & D_i \geq x \geq \overline{D_i}, \\ 0, & \text{otherwise}, \end{cases} \tag{1.33}$$

where the demand is $\widetilde{D_i} = (\underline{D_i}, D_i, \overline{D_i})$. The fuzzy objective function is complex and the concavity proof is difficult; therefore, Dutta and Chakraborty developed an algorithm to find the optimal order quantity. They defined four situations of demand in relation to the Q^* and run the complete procedure for each one of these. They ran some numerical examples to provide validation for their method and made recommendations for further research to include salvage value and holding cost as well as two-way substitution.

Considering a stylized scenario for two products, without a loss of generality, assume that product 1 substitutes for product 2 one-to-one, and if there is a substitution, this item is sold at the price of product 2. We also assume that the selling price of the substituted item is higher than the cost of the substitute as well as its salvage value. Then the actual end of period profit for the buyer is:

$$\begin{array}{ll} \textit{Case 1.}\ \sum_{i=1}^{2} [p_i x_i - v_i Q_i + g_i(Q_i - x_i)] & \text{if } x_1 \leq Q_1; x_2 \leq Q_2 \\ \textit{Case 2.}\ p_1 Q_1 + p_2 x_2 - \sum_{i=1}^{2} v_i Q_i + g_2(Q_2 - x_2) - B_1(x_1 - Q_1) & \text{if } x_1 > Q1; x_2 \leq Q_2 \\ \textit{Case 3.}\ p_1 x_1 + p_2 Q_2 - \sum_{i=1}^{2} v_i Q_i + p_2 Min(x2 - Q_2, Q_1 - x_1) & \text{if } x_1 \leq Q1; x_2 > Q_2 \\ \quad + g_1[Q_1 - x_1 - (x_2 - Q_2)]^+ - B_2[x_2 - Q_2 - (Q_1 - x_1)]^+ & \\ \textit{Case 4.}\ \sum_{i=1}^{2} p_i Q_i - v_i Q_i - B_i(x_i - Q_i) & \text{if } x_1 > Q1; x_2 > Q_2 \end{array} \tag{1.34}$$

and based on this, the expected profit function is:

$$\begin{aligned} E[\pi(Q_1, Q_2)] = E\Big[& [p_1 Min(x_1, Q_1) + p_2 Min[x_2, Q_2 + (Q_1 - x_1)^+] - v_1 Q_1 - v_2 Q_2 \\ & + g_1[Q_1 - x_1 - (x_2 - Q_2)^+]^+ + g_2(Q_2 - x_2)^+ - B_1(x_1 - Q_1)^+ \\ & - B_2[x_2 - Q_2 - (Q_1 - x_1)^+]^+ \Big] \\ = p_1 & \left[\int_0^{Q_1} x_1 f_1(x_1) dx_1 + \int_{Q_1}^{\infty} Q_1 f_1(x_1) \mathrm{d}x_1 \right] \\ & + p_2 \left[\int_0^{Q_2} x_2 f_2(x_2) \mathrm{d}x_2 + \int_{Q_2}^{\infty} Q_2 f_2(x_2) \mathrm{d}x_2 \right. \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{Q_1} \int_{Q_2}^{\infty} (Q_1 - x_1) f(x_1, x_2) \mathrm{d}x_2 \mathrm{d}x_1 \Bigg] \\
&- v_1 Q_1 - v_2 Q_2 + g_1 \left[\int_0^{Q_1} \int_0^{Q_2} (Q_1 - x_1) f(x_1, x_2) \mathrm{d}x_2 \mathrm{d}x_1 \right. \\
&\left. + \int_0^{Q_1+Q_2-x_2} \int_{Q_2}^{Q_1+Q_2} (Q_1 + Q_2 - x_1 - x_2) f(x_1, x_2) \mathrm{d}x_2 \mathrm{d}x_1 \right] \\
&+ g_2 \int_0^{Q_2} (Q_2 - x_2) f_2(x_2) \mathrm{d}x_2 - B_1 \int_{Q_1}^{\infty} (x_1 - Q_1) f_1(x_1) \mathrm{d}x_1 \\
&- B_2 \left[\int_{Q_1}^{\infty} \int_{Q_2}^{\infty} (x_2 - Q_2) f_1(x_1, x_2) \mathrm{d}x_2 \mathrm{d}x_1 \right. \\
&\left. + \int_0^{Q_1} \int_{Q_1+Q_2-x_1}^{\infty} (x_1 + x_2 - Q_1 - Q_2) f(x_1, x_2) \mathrm{d}x_2 \mathrm{d}x_1 \right]. \quad (1.35)
\end{aligned}
$$

Cai et al. (2004) used a similar expected profit function as above and proved that it is concave and submodular. Using this property, the optimal order quantities can be found by setting the derivatives with respect to Q_1 and Q_2 equal to zero. If we define $G(Q_1, Q_2) = \int_0^{Q_1} \int_0^{Q_1+Q_2-x_1} f(x_1, x_2) \mathrm{d}x_2 \mathrm{d}x_1$, the following holds:

$$
F_1(Q_1^*) + \frac{(p_2 + B_2) - g_1)}{(p_1 + B_1) - (p_2 + B_2)} G(Q_1^*, Q_2^*) = \frac{(p_1 + B_1) - v_1}{(p_1 + B_1) - (p_2 + B_2)}, \quad (1.36)
$$

$$
F_2(Q_2^*) + \frac{(p_2 + B_2) - g_1)}{(p_2 + B_2) - g_2)} \left[G(Q_1^*, Q_2^*) - F(Q_1^*, Q_2^*) \right] = \frac{(p_2 + B_2) - v_1}{(p_2 + B_2) - g_2}. \quad (1.37)
$$

$F_i(Q_i^*)$ represents the probability of all of the demand for item i being satisfied when the stock level is Q_i^*. $G(Q_1^*, Q_2^*)$ is the probability that the total demand is satisfied given that item 1 was substituted for item 2. $F(Q_1^*, Q_2^*) = \int_0^{Q_1} \int_0^{Q_2} f(x_1, x_2) \mathrm{d}x_2 \mathrm{d}x_1$ is defined as the probability that the demand for each item is satisfied without any substitution. Finally, $F_2(Q_2^*) + G(Q_1^*, Q_2^*) - F(Q_1^*, Q_2^*)$ is the probability that all of the demand for item 2 is satisfied using either of the items. Cai et al proved four different properties of the optimal order quantities. Property 1 shows that as the unit price of item i increases, Q_1^* decreases and Q_2^* increases and, evidently Q_2^* decreases as the unit price of item 2 increases. Property 2 states that when the price of each item increases, their respective optimal quantities decrease. Conversely, the increase in price of item 1 decreases the optimal quantity for item 2. Property 3 states a similar argument related to salvage cost. Property 4 indicates that the optimal order quantity of each item is linearly related to their respective mean demands. Property 5 states that the variance of item i affects the optimal quantity of item j reversely. In this paper, the authors showed that the expected profits and the fill rate can be improved by using substitution.

Table 1.1 Notations for two-way substitution

Symbol	Meaning
R	Review period
L_i	Replenishment lead time
$L_i + R_i$	Replenishment cycle
$f_{x^i}(x_0)$	Density function of demand over the replenishment cycle for product i
β_i	Parameter that satisfies, $\sigma_i = \beta_i \sigma_1$
r	Inventory holding cost
S_i	Order up-to level
K_i	Safety factor, $S_i = \overline{x_i} + K_i \sigma_i$
$0 < \alpha_{ij} < 1$	Probability that a costumer will substitute j for a unit of i
$f_u(u_0)$	Density function of standard normal distribution
$G_u(k)$	Tabulated function of the standard normal distribution

1.3.2 Two-Way Substitution

Unlike the one-way substitution, in two-way substitution case, each of the items can be used to supply the demand for another one. This only occurs when the demand for one item is higher than the quantity ordered and the demand for the substitute item is lower than the quantity ordered. McGillivray and Silver (1978) and Parlar and Goyal (1984) assumed whenever substitution is possible, there is a probability that a customer will accept a substitute product. In Parlar's case, this probability was between 0 and 1, whereas it was fixed for McGillivray. In McGillivray's paper, the demand, x_i , is assumed to be normally distributed with a mean of $\bar{x}_i$ and a standard deviation of $\sigma_i = \beta_i \sigma 1$. The order up to level is given as $S_i = \bar{x}_i + K_i \sigma_i$ and the expected shortage per replenishment cycle is $ESPRC_i = \sigma_i G_u(K_i)$. The unit variable costs and shortage cost of the substitutable, items are also assumed to be identical. This assumption is justified by the fact that in reality when two items are substitutable they will have similar prices. Different levels of substitutability were considered in the paper. The notation used in their paper is shown on Table 1.1.

We know that $G_u(k) = \int_k^\infty (u_0 - k) f_u(u_0) \mathrm{d}u_0$, and $\frac{\mathrm{d}G_u(K_i)}{\mathrm{d}K_i} = -P_{u\geq}(K_i)$. By setting the partial derivative of ETRC with respect to K_i to 0, we find that $P_{u\geq}(K_i*) = \frac{Rvr}{B}$ for $i = 1, \ldots, N$. Using the standard normal property $f_u(u_0) = P_{u\geq}(u_0) + G_u(u_0)$, ETRC can be reduced to:

$$ETRC(K_1^*, K_2^*, .., K_n^*) = \frac{1}{2} DR^2 vr + \sigma_1 B f_u(K^*) \sum_{i=1}^{N} \beta_i. \tag{1.38}$$

If we assume that there is full demand transferability and all items are perfect substitutes of each other, $a_{ij} = 1$, a shortage happens only when the total demand for all items is smaller than the total stock up-to level. The total shortage and on-hand inventory decrease the same amount by the transfer sales; therefore, the total net stock stays the same as the general case. Consequently, the expected total relevant costs with perfect substitution is:

$$ETRC_t(K_{t1}^*, K_{t2}^*, .., K_{tn}^*) = \frac{1}{2}DR^2vr + \sigma_1 B\sqrt{\sum \beta_i^2} f_u(K^*). \quad (1.39)$$

The minimum cost equation is the same as the single item newsvendor model with the demand equivalent to the total demand of substitutable item problem. The savings from the substitutability can be expressed as $ETRC(K_1^*, K_2^*, .., K_n^*) - ETRC_t(K_{t1}^*, K_{t2}^*, .., K_{tn}^*)$. Given (1.38) and (1.39) , the maximum possible savings will occur when substitutability, a_{ij}, is equal to 1. Thus,

$$MPS = \sigma_1 B\left(\sum \beta_i - \sqrt{\sum \beta_i^2}\right) f_u(K^*)), \text{ and } K_1^* = ... = K_N^* = K^* = \frac{\sum \beta_i K_{ti}^*}{\sqrt{\sum \beta_i^2}}. \quad (1.40)$$

In addition to these results, the authors (through a numerical analysis for a two item model) show that when both items are substitutable to each other, the potential savings increase when K^* increases. In the case of one way substitution, where $a_{12} = 0$ and $a_{21} = 1$, the optimal policy is to stock item 1 only. This theorem also holds when $0 < a_{12} < 1$ and $a_{21} = 1$. There is no analytic expression for *ESPRC* when both items are partially substitutable to each other. Hence, McGillivray and Silver simulated a two-item inventory problem with substitutability. As a result, they demonstrated that when substitutability levels are between 0 and 0.75, the model acts as an independent item inventory control problem. Furthermore, a cost penalty larger than 20% of the MPS only occurs when one of the items is a perfect substitute of the other. For the case of partial substitutability, a heuristic approach was developed and tested.

In relation to the heuristic approach, the expected transferred demands were defined as $E(T_{21}) = a_{21}ESPRC_2$ and $E(T_{12}) = a_{12}ESPRC_1$ for items 1 and 2, respectively. This approach tries to find the optimal values for K and S using $P_{u\geq}(K_i^*) = Rvr/[B(1-a_{ji}) + a_{ji}Rvr]$ and $S_i = [\bar{x}_i + a_{ji}\sigma_j G_u(K_j)] + K_i\sigma_i$ for $i \neq j$ $i = 1, \ldots, N$. This two-item model can also be extended to include multiple items and it is computationally straightforward.

Netessine and Rudi's paper Netessine and Rudi (2003) examines the optimal inventory stocking policies for a given product line under the notion that consumers who do not find their first-choice product in the current inventory might substitute a similar product for it (consumer-driven substitution). Namely, there is an arbitrary number of products and each consumer has a first choice product. If this product is out of stock, the consumer might choose one of the other products as a substitute.

Let α_{ij} denote the probability that a customer will substitute j for a unit of i. The demand vector, $D = (D_1, ..., D_n)$, follows a known continuous multivariate demand distribution with positive support. In the centralized inventory model, the expected profit of the company who manages n products is:

$$\pi = E\sum_i \left[p_i \min\left(D_i + \sum_{j\neq i} \alpha_{ij}(D_j - Q_j)^+, Q_i \right) - v_i Q_i + g_i \left(Q_i - \left(D_i + \sum_{j\neq i} \alpha_{ij}(D_i - Q_j)^+ \right) \right)^+ \right].$$

The demand vector for i is $D_i^S = D_i + \sum_{j\neq i} \alpha_{ij}(D_j - Q_j)^+$, where the superscript S indicates that the effect from substitution has been accounted for. In other words, D_i^S is the sum of the first-choice demand and demand from substitution. It is conventional to define $u_i = p_i - v_i$, the unit underage cost; and $o_i = v_i - g_i$,the unit overage cost. This paper proves that the first-order necessary optimality conditions of the centralized problem are given by:

$$Pr(D_i < Q_i^c) - Pr(D_i < Q_i^C < D_i^S) + \sum_{j\neq i} \frac{u_i + o_j}{u_i + o_i} \alpha_{ij} Pr(D_j^S < Q_i^C, D_i > Q_i^C) = \frac{u_i}{u_i + o_i}.$$

In the decentralized inventory model, the profit for each firm i is:

$$\pi_i = E\left[u_i D_i^S - u_i(D_i^S - Q_i)^+ - o_i\left(Q_i - D_i^S\right)\right], i = 1, \ldots n.$$

This paper also shows that any Nash equilibrium is characterized by the following optimality conditions:

$$Pr(D_i < Q_i^d) - Pr(D_i < D_i^d < D_i^S) = \frac{u_i}{u_i + o_i}.$$

After comparing the optimal ordering quantity Q_i^c and Q_i^d, the paper finds that: there exist situations when $Q_i^c \geq Q_i^d$ for some i, it is always true that $Q_i^c \leq Q_i^d$ for at least one i, suppose that all the costs are independent and identically distributed, and the consumers are equally likely to switch to any of the $(N-1)$ products for all i, j. Then $Q_i^c \leq Q_i^d$ for all i.

Nagarajan and Rajagopalan (2008) took a different approach and assumed the demands of products to be correlated. They defined the total demand to be D, and the demand portions of the products to be p, $(1-p)$. Without loss of generality, D is set to be 1, and the optimal order quantities for product 1 and 2 differ from the general newsvendor solution by $(1-\gamma)$. This indicates that the higher the fraction of substitution, the lower the inventory levels. In the case of asymmetric costs and random total demand, a fixed proportion, γ_i, of the customers looking for item i when it is depleted will purchase the substitute and $(1-\gamma_i)$ of them will not make a purchase. If we let $\gamma_i^* = \max\{(p_i + B_i) - (h_i + 2c_i)/(p_j + h_j + B_i), 1\}, i, j = 1, 2, i \neq j$ then, if $\gamma_i \leq \gamma_i^*$, the "partially decoupled" inventory policy is optimal. It is evident that when product 2 is priced higher, the optimal base stock for product 1 is lower. Especially, in the case of high enough p_2 and h_2, this base stock level can be below the mean or even close to zero. This means that the risk-pooling effect of substitution reduces the inventories of both products. This effect is more apparent for the inventory of the lower priced product. The authors show that this method can easily be applied to the n-product and multi-period model.

Focusing on a stylized two-product setting, the end of period profit for the buyer are:

$$\begin{array}{lll}
\textit{Case 1.} & \sum_{i=1}^{2}[p_i x_i - v_i Q_i + g_i(Q_i - x_i) & \text{if } x_1 \le Q_1; x_2 \le Q_2 \\
\textit{Case 2.} & p_1 Q_1 + p_2 x_2 - \sum_{i=1}^{2} v_i Q_i + p_1 \mathrm{Min}(x_1 - Q_1, Q_2 - x_2) & \text{if } x_1 > Q1; x_2 \le Q_2 \\
& +g_2[(Q_2 - x_2) - (x_1 - Q_1)]^+ - B_1[(x_1 - Q_1) - (Q_2 - x_2)]^+ & \\
\textit{Case 3.} & p_1 x_1 + p_2 Q_2 - \sum_{i=1}^{2} v_i Q_i + p_2 \mathrm{Min}(x2 - Q_2, Q_1 - x_1) & \text{if } x_1 \le Q1; x_2 > Q_2 \\
& +g_1[(Q_1 - x_1) - (x_2 - Q_2)]^+ - B_2[(x_2 - Q_2) - (Q_1 - x_1)]^+ & \\
\textit{Case 4.} & \sum_{i=1}^{2} p_i Q_i - v_i Q_i - B_i(x_i - Q_i) & \text{if } x_1 > Q1; x_2 > Q_2
\end{array} \tag{1.41}$$

and based on this, the expected profit function is:

$$\begin{aligned}
E[\pi(Q_1,Q_2)] = E\Big[& [p_1 Min(x_1,Q_1) + p_2 Min[x_2, Q_2 + (Q_1 - x_1)^+] - v_1 Q_1 - v_2 Q_2 \\
& + g_1[Q_1 - x_1 - (x_2 - Q_2)^+]^+ + g_2(Q_2 - x_2)^+ - B_1(x_1 - Q_1)^+ \\
& - B_2[x_2 - Q_2 - (Q_1 - x_1)^+]^+ \Big] \\
= p_1 \Big[& \int_0^{Q_1} x_1 f_1(x_1)\mathrm{d}x_1 + \int_{Q_1}^{\infty} Q_1 f_1(x_1)\mathrm{d}x_1 \\
& + \int_0^{Q_1}\int_{Q_2}^{\infty}(Q_2 - x_2) f(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1 \Big] \\
+ p_2 \Big[& \int_0^{Q_2} x_2 f_2(x_2)\mathrm{d}x_2 + \int_{Q_2}^{\infty} Q_2 f_2(x_2)\mathrm{d}x_2 \\
& + \int_0^{Q_1}\int_{Q_2}^{\infty}(Q_1 - x_1) f(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1 \Big] \\
- v_1 Q_1 - v_2 Q_2 + g_1 \Big[& \int_0^{Q_1}\int_0^{Q_2}(Q_1 - x_1) f(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 \\
& + \int_0^{Q_1+Q_2-x_2}\int_{Q_2}^{Q_1+Q_2}(Q_1 + Q_2 - x_1 - x_2) f(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 \Big] \\
+ g_2 \Big[& \int_0^{Q_1}\int_0^{Q_2}(Q_2 - x_2) f(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{0}^{Q_1+Q_2-x_1} \int_{Q_1}^{Q_1+Q_2} (Q_1+Q_2-x_1-x_2) f(x_1,x_2) \mathrm{d}x_2 \mathrm{d}x_1 \Bigg] \\
&- B_1 \Bigg[\int_{Q_1}^{\infty} \int_{Q_2}^{\infty} (x_1-Q_1) f(x_1,x_2) \mathrm{d}x_1 \mathrm{d}x_2 \\
&\quad + \int_{Q_1+Q_2-x_2}^{\infty} \int_{0}^{Q_2} (x_1+x_2-Q_1-Q_2) f(x_1,x_2) \mathrm{d}x_2 \mathrm{d}x_1 \Bigg] \\
&- B_2 \Bigg[\int_{Q_1}^{\infty} \int_{Q_2}^{\infty} (x_2-Q_2) f(x_1,x_2) \mathrm{d}x_2 \mathrm{d}x_1 \\
&\quad + \int_{0}^{Q_1} \int_{Q_1+Q_2-x_1}^{\infty} (x_1+x_2-Q_1-Q_2) f(x_1,x_2) \mathrm{d}x_2 \mathrm{d}x_1 \Bigg].
\end{aligned} \tag{1.42}
$$

Pasternack and Drezner (1991) proved that this function is concave and showed that the optimal quantities can be found using a specific distribution and parameters. Assuming that the revenue from substitution is different from the revenue from regular sales, the authors also explored the effect of substitution on the order quantities and showed that for a given revenue of t_2 for each product 2 that is substituted for product 1:

$$\frac{\mathrm{d}Q_1^*}{\mathrm{d}t_1} > 0 \text{ and } \frac{\mathrm{d}Q_2^*}{\mathrm{d}t_1} < 0, \tag{1.43}$$

and a similar result holds for product 1. This result implies that if the revenue from substitution of one product increases, the optimal order quantity for the other product will decrease and the substitute product will increase. The authors analytically solved the case for the one-way substitution and reached similar insights. In addition, the authors explored the effect of substitution on the total inventory levels and observed that when the revenue from substitution increases, the optimal quantity of the substitutable product increases faster than the substitute.

Rajaram and Tang (2001) studied the same problem but allowed the substitution parameter to be anywhere between 0 and 1. The heuristic they presented explores how the demand variation and correlation as well as the substitution affect the expected order quantities and expected profits. Khouja et al. (1996) used Monte Carlo simulation to find the optimal order quantities. Six events are defined to represent this model. First event is when the demand for each item is less than its order quantities. Second event is when the demand for each item is equal to or higher than its order quantities. The third and the fourth events are when the demand for item 1 is greater than the order quantity, and the excess quantity of item j is sufficient or insufficient, respectively. Similar case holds for the fifth and the sixth events. They define the upper and lower quantity bounds for each item and prove that the optimal quantities will be between these two values. The first property, which aids the proof of Lemma 1, states that it is more profitable to sell customers one unit of i than to sell t_j quantity of item j. They define the lower bounds to be Q_1^{L} and Q_2^{L}, where $F_1(X_1 = Q_1^{\mathrm{L}}) \approx 1$ and $F_2(X_2 = Q_2^{\mathrm{L}}) \approx 1$ holds. Lemma 1 indicates that the optimal solution will always be higher than the lower bound. In order to prove this, three scenarios that violate lemma 1 are considered. They show that for each of the cases, the expected profit increases when the solution is equal to or higher than the lower bound. They define upper bounds to be Q_1^{U} and Q_2^{U}, where $F_3(X_3 = Q_1^{\mathrm{U}}) \approx 1$ and $F_4(X_4 = Q_2^{\mathrm{U}}) \approx 1$, $X_3 = X_1 + t_1(X_2 - Q_2^{\mathrm{L}})$ and $X_4 = X_2 + t_2(X_1 - Q_1^{\mathrm{L}})$. Using similar arguments to lemma 1, they prove that the optimal solution is lower than the upper bound. Stricter upper bounds can be found assuming X_1 and X_2 are normally distributed; consequently, X_3 and X_4 can be assumed to be normally distributed. They prove that any optimal solution will be less than Q_1^{N} and Q_2^{N} , where Q_1^{N} and Q_2^{N} are the solutions to the newsvendor problems with demands X_3 and X_4. Numerical tests were run to gain insights to the problem. As a result, it was found that as t_1 increases, Q_1 increases and Q_2 decreases. This can be explained by the decrease in the effective cost of underestimating item 2. Consequently, the demand for item 1 increases and the demand for item 2 decreases.

The assumptions for this paper are: (1) demand is normally distributed, (2) demand for different items is independent, (3) the salvage value for unsold items is zero, (4) there is no penalty for unmet demand, (5) price, cost, average demand, and standard deviation of demand for all products are the same, (6) for the model with budget constraint, only two items are considered, (7) analyzing substitutability, only a two-item newsvendor problem is considered, and it is assumed that the customer takes a single unit of the substitution product, and (8) substitutability is assumed to be symmetric.

In the model without additional complications, it is shown that the value of product-mix information increases with the number of items, whereas the value of global information decreases with the number of items. The value of both product-mix information and global information decreases with a budget constraint. Furthermore, the value of perfect information also decreases with a budget constraint. The probability of substitution decreases the value of product-mix information such that it is zero with complete substitution, and increases the value of global information so that it is equal to the value of perfect information with complete substitution.

1.4 Extensions

In this final section, we describe two recent extensions for handling the unconstrained MPNP for the case of substitute products. The first extension examines the case where the demand is price dependent. This is a common situation that arises in practices that customers substitute a different product when the price of the desired product has increased. For example, the supermarkets stock two different brand shampoos with same price and similar quality. If one of the products has increased their price, the price sensitive customers will choose the product with the lower price. The second case addresses the situation where demand is quantity dependent. This is reasonable in reality because an increase in shelf space for a product attracts more customers to buy it due to its visibility and popularity. Conversely, low stocks of certain goods (e.g., perishable food) might leave the impression that they are not fresh. In both cases, we present the results for a stylized scenario for two products.

1.4.1 Price Linear Demand

Carrillo et al. (2011) analyze the stocking decision under price linear demand for substitutive products. The demand function for product $i (i = 1,2)$ is:

$$x_i = a_i - b_i p_i + s p_j + \varepsilon_i, \tag{1.44}$$

where a_i is the market share for product i, b_i represents the price elasticity of demand for product i, and s is the symmetric price-based substitution effect parameter. ε_i is defined as a continuous random variable with probability density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$ in the range of $[-d_i, d_i]$ with mean μ_i The profit for each product is:

$$\pi_i(Q_i, x_i) = \begin{cases} p_i x_i - v_i Q_i + g_i(Q_i - x_i) & \text{if } Q_i \geq x_i \\ p_i Q_i - v_i Q_i - B_i(x_i - Q_i) & \text{if } Q_i < x_i \end{cases}. \tag{1.45}$$

Let $z_i = Q_i - a_i + b_i p_i$, the expected profit for each product i is:

$$\begin{aligned} E[\Pi(z_1, z_2, p_1, p_2)] = & \left\{ \int_{-d_1}^{z_1} [p_1(a_1 - b_1 p_1 + s p_2 + \varepsilon_1) + g_1(z_1 - \varepsilon_1)] f(\varepsilon_1) \mathrm{d}\varepsilon_1 \right\} \\ & + \left\{ \int_{z_1}^{d_1} [p_1(a_1 - b_1 p_1 + s p_2 + z_1) + B_1(z_1 - \varepsilon_1)] f(\varepsilon_1) \mathrm{d}\varepsilon_1 \right\} \\ & - v_1(a_1 - b_1 p_1 + s p_2 + z_1) \\ & + \left\{ \int_{-d_2}^{z_2} [p_2(a_2 - b_2 p_2 + s p_1 + \varepsilon_2) + g_2(z_2 - \varepsilon_2)] f(\varepsilon_2) \mathrm{d}\varepsilon_2 \right\} \\ & + \left\{ \int_{z_2}^{d_2} [p_2(a_2 - b_2 p_2 + s p_1 + z_2) + B_2(z_2 - \varepsilon_2)] f(\varepsilon_2) \mathrm{d}\varepsilon_2 \right\} \\ & - v_2(a_2 - b_2 p_2 + s p_1 + z_2) \\ = & \sum_{i=1}^{2} \left\{ (p_i - v_i)(a_i - b_i p_i) - (v_i - g_i) z_i \right. \\ & \left. - (p_i - g_i) \left[\int_{z_i}^{d_i} (\varepsilon_i - z_i) f(\varepsilon_i) \mathrm{d}\varepsilon_i - \mu_i \right] \right\} + B_i \int_{z_i}^{d_i} (z_i - \varepsilon_i) \\ & \times f(\varepsilon_i) \mathrm{d}\varepsilon_i + s p_1(p_2 - v_2) + s p_2(p_1 - v_1). \end{aligned} \tag{1.46}$$

The FOCs are:

$$\frac{\partial E[\Pi]}{\partial z_1} = -v_1 + g_1 F(z_1) + (p_1 + B_1)[1 - F(z_1)], \tag{1.47}$$

$$\frac{\partial E[\Pi]}{\partial z_2} = -v_2 + g_2 + (p_2 + B_2)[1 - F(z_2)], \tag{1.48}$$

$$\frac{\partial E[\Pi]}{\partial p_1} = 2b_1\left[\frac{a_1 + b_1 v_1}{2b_1} - p_1\right] - \int_{z_1}^{d_1} (\varepsilon_1 - z_1) f(\varepsilon_1) \mathrm{d}\varepsilon_1 + 2sp_2 - sv_2 + \mu_1, \tag{1.49}$$

$$\frac{\partial E[\Pi]}{\partial p_2} = 2b_2\left[\frac{a_2 + b_2 v_2}{2b_2} - p_2\right] - \int_{z_2}^{d_2} (\varepsilon_2 - z_2) f(\varepsilon_2) \mathrm{d}\varepsilon_2 + 2sp_1 - sv_1 + \mu_2. \tag{1.50}$$

The second-order conditions are:

$$\frac{\partial^2 E[\Pi]}{\partial {z_i}^2} = (g_i - p_i - B_i) f(z_i) \qquad \text{for } i = 1, 2, \tag{1.51}$$

$$\frac{\partial^2 E[\Pi]}{\partial {p_i}^2} = -2b_i \qquad \text{for } i = 1, 2, \tag{1.52}$$

$$\frac{\partial^2 E[\Pi]}{\partial z_1 \partial p_1} = 1 - F(z_1), \tag{1.53}$$

$$\frac{\partial^2 E[\Pi]}{\partial z_1 \partial p_2} = 0, \tag{1.54}$$

$$\frac{\partial^2 E[\Pi]}{\partial z_2 \partial p_2} = 1 - F(z_2), \tag{1.55}$$

$$\frac{\partial^2 E[\Pi]}{\partial z_2 \partial p_1} = 0, \tag{1.56}$$

$$\frac{\partial^2 E[\Pi]}{\partial p_1 \partial p_2} = 2s. \tag{1.57}$$

We can't prove that the Hessian is strictly concave without the specific value for parameters.

For specific values of z_1 and z_2, (1.49) and (1.50) are strictly and jointly concave in p_1 and p_2. Since it was assumed that $b_j > s$ for $j = 1, 2.$, (1.52) and (1.57) indicate that $|H_1| < 0$ and $|H_2| = 4b_1 b_2 - 4s^2 > 0$. Thus, for given values of z_1 and z_2, the optimal prices can be determined by solving the following simultaneous equations (obtained by setting the FOCs in (1.49) and (1.50) equal to 0):

$$-2b_1 p_1 + 2sp_2 = \int_{z_1}^{d_1} (\varepsilon_1 - z_1) f(\varepsilon_1) \mathrm{d}\varepsilon_1 + sv_2 - (a_1 + b_1 v_1) - \mu_1, \tag{1.58}$$

$$2sp_1 - 2b_2 p_2 = \int_{z_2}^{d_2} (\varepsilon_2 - z_2) f(\varepsilon_2) \mathrm{d}\varepsilon_2 + sv_1 - (a_2 + b_2 v_2) - \mu_2. \tag{1.59}$$

The solution to this set of equations is:

$$p_1(z_1,z_2) = \frac{b_2u_1+su_2}{2(b_1b_2-s^2)}, \tag{1.60}$$

$$p_2(z_1,z_2) = \frac{b_1u_2+su_1}{2(b_1b_2-s^2)}, \tag{1.61}$$

where $u_1 = (a_1+b_1c_1) - \int_{z_1}^{d_1}(\varepsilon_1 - z_1)f(\varepsilon_1)\mathrm{d}\varepsilon_1 - sc_2 + \mu_1$ and $u_2 = (a_2+b_2c_2) - \int_{z_2}^{d_2}(\varepsilon_2 - z_2)f(\varepsilon_2)\mathrm{d}\varepsilon_2 - sc_1 + \mu_2$.

Then the following algorithm could determine the optimal prices, stocking quantities, and the corresponding optimal profit:

1. Set $z_1 = -d_1 - 0.01$; $z_2 = -d_2 - 0.01$; $Profit = 0$; $m_1 = m_2 = 0$, $p_{1t} = p_1 = 0$; $p_{2t} = p_2 = 0$ and $z_{1t} = z_{2t} = 0$.
2. $z_1 = z_1 + 0.01$. If $z_1 > d_1$, go to 6.
3. $z_2 = z_2 + 0.01$. If $z_2 > d_2$, go to 2.
4. Compute $p_1(z_1,z_2)$ using (1.60) and $p_2(z_1,z_2)$ using (1.61). Set $p_{1t} = p_1(z_1,z_2)$ and $p_{2t} = p_2(z_1,z_2)$.
5. Compute $E[\Pi(z_1,z_2,p_{1t},p_{2t}]$ using (1.46). If $Profit > E[\Pi(z_1,z_2,p_{1t},p_{2t})]$, go to 3, else set $Profit = E[\Pi(z_1,z_2,p_{1t},p_{2t})]$; $z_{1t} = z_1$; $z_{2t} = z_2$; $m_1 = p_{1t}$, $m_2 = p_{2t}$ and go to 3.
6. The optimal market prices are: $p_1* = m_1$ and $p_2* = m_2$; the optimal stocking quantities are: $q_1* = a_1 - b_1m_1 + sm_2 + z_{1t}$, and $q_2* = a_2 - b_2m_2 + sm_1 + z_{2t}$ and associated optimal profit is $Profit$.

1.4.2 Quantity Linear Demand

The demand function for quantity linear demand of product i ($i = 1,2$) is

$$x_i = a_i + b_iq_i - sq_j + \varepsilon_i, \tag{1.62}$$

where a_i represents the relative market share for product i, b_i is the quantity elasticity of demand, and s is the symmetric quantity-based substitution effect parameter. Also ε_i is defined as a continuous random variable with probability density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$ in the range of $[-d_i,d_i]$ with mean μ_i. The profit for each product is:

$$E[\Pi_i(z_1,z_2,q_1,q_2)] = \left\{\int_{-d_i}^{z_i}[p_i(a_i+b_iq_i-sp_j+\varepsilon_i)+g_i(z_i-\varepsilon_i)]f(\varepsilon_i)\mathrm{d}\varepsilon_i\right\} + \left\{\int_{z_i}^{d_i}[p_i(a_i+b_iq_i-sq_j+z_i)+B_i(z_i-\varepsilon_i)]f(\varepsilon_i)\mathrm{d}\varepsilon_i\right\}$$

$$
\begin{aligned}
&-v_i(a_i + b_i q_i - s q_j + z_i)\\
&= (p_i - v_i)(a_i + b_i q_i - s q_j) - (v_i - g_i) z_i + (p_i - g_i)\mu_i\\
&-(p_i - g_i + B_i)\int_{z_i}^{d_i} (\varepsilon_i - z_i) f(\varepsilon_i)\mathrm{d}\varepsilon_i, \qquad (1.63)
\end{aligned}
$$

where $z_i = q_i - (a_i + b_i q_i - s q_j)$. The total profit is $E[\Pi(z_1, z_2, q_1, q_2)] = E[\Pi_1)] + E[\Pi_2]$

The FOCs are:

$$
\begin{aligned}
\frac{\partial E[\Pi]}{\partial q_i} &= (p_i - g_i)b_i + (g_i - v_i) + (p_i - g_i + B_i)[1 - F(z_i)](1 - b_i) +\\
&\quad - s\{(p_j - g_j) + (p_j - g_j + B_j)[1 - F(z_j)]\}. \qquad (1.64)
\end{aligned}
$$

The second-order conditions are:

$$
\frac{\partial^2 E[\Pi]}{\partial {q_i}^2} = -(p_i - g_i + B_i)(1 - b_i)^2 f(z_i) - (p_j - g_j + B_j)s^2 f(z_j), \qquad (1.65)
$$

$$
\frac{\partial^2 E[\Pi]}{\partial q_1 \partial q_2} = -(p_1 - g_1 + B_1)(1 - b_1)s f(z_1) - (p_2 - g_2 + B_2)(1 - b_2)s f(z_2). \quad (1.66)
$$

From the Hessian Matrix, $|H_1| < 0$, $|H_2| = u_1 u_2[(1 - b_1)(1 - b_2) - s^2]^2 > 0$, where $u_i = (p_i - g_i + B_i)f(y_i)$. Since the objective function is strictly concave, we can obtain the solution for this problem from the FOCs (i.e., by setting them equal to 0 and simultaneously solving for the decision variables).

1.5 Conclusions and Directions for Future Research

In this paper, we have reviewed and critiqued the literature to date for the MPNP. As is obvious, the majority of prior research has focused on determining the optimal stocking policy for the constrained MPNP. More recent work on exploring the impact of substitutability has also been undertaken and in this setting, we present two possible extensions of the MPNP for price and quantity substitution effects. For both these cases, we show that optimal solutions can be obtained through either a search process (for price substitution) or a structural analysis (for quantity substitution). Here we point out a few areas where further research is needed.

1.5.1 Price-Dependent Demand

Previous researches on MPNP assume the independence of price and market demand. Recent work has mitigated this issue by addressing the joint ordering and pricing problem in the MPNP framework. But most of these works only consider single budget constraint, while in practice the retailer may face multiple resource constraints. So it would be interesting to extend the study to consider the problem with multiple constraints. Due to the complexity of the problem, high quality heuristics procedures are anticipated to find good solutions.

1.5.2 Multiple Suppliers

Nearly all the models in this chapter assume single supplier. However, in practice, retailers may face several suppliers when making the merchandise decision. It would be interesting to incorporate multiple suppliers into MPNP, especially under price competition between potential suppliers and availability of several supply options. We see many opportunities for future research to help bridge this gap.

1.5.3 Product Substitution

Incorporating the substitution effects can have a significant effect on profitability. However, most previous studies on the substitution effects of the MPNP only focus on two products substitutability. It would be interesting to extend the analysis in a more generalized case. This extension requires a better understanding of interdependencies among the demands for related products. So an empirical investigation of the generalized substitution effects in customer decision making will also be an attractive future research area.

1.5.4 Risk and Hedging

The classical newsvendor problem is based on the assumption that most of the supply chains are risk neutral. The research on risk-averse supply chains has been considered by several authors. However, these papers focused on independent demand. As an extension, price and quantity-dependent demands can be considered. The hedging problem has been tackled by Vaagen and Wallace in the fashion industry, this research could be extended to other industries with different sales behavior. Also, the pricing strategies for these hedging portfolios could be examined to identify policies that further reduce profit risk.

References

Abdel-Malek, L., & Areeratchakul, N. (2007). A quadratic programming approach to the multi-product newsvendor problem with side constraints. *European Journal of Operational Research*, *176*(3), 1607–1619.

Abdel-Malek, L., & Montanari, R. (2005a). An analysis of the multi-product newsboy problem with a budget constraint. *International Journal of Production Economics*, *97*(3), 296–307.

Abdel-Malek, L., & Montanari, R. (2005b). On the multi-product newsboy problem with two constraints. *Computers & Operations Research*, 32, 2095–2116.

Abdel-Malek, L. Montanari, R., & Morales, L. (2004). Exact, approximate, and generic iterative models for the multi-product newsboy problem with budget constraint. *International Journal of Production Economics*, *91*(2), 189–198.

Anupindi, R., Dada, M., & Gupta, S. (1998). Estimation of consumer demand with stock-out based substitution: an application to vending machine products. *Marketing Science*, *17*(4), 406–423.

Areeratchakul, N., & Abdel-Malek, L. (2006). An approach for solving the multi-product newsboy problem. *International Journal of Operational Research*, *3*(3), 219–227.

Arrow, K., Harris, T., & Marshak, J. (1951). Optimal inventory policy. *Econometrica*, *19*(3), 250–272.

Aviv, Y., & Federgruen, A. (2001). Capacitated multi-item inventory systems with random and seasonally fluctuating demands: implications for postponement strategies. *Management Science*, *47*(4), 512–531.

Bassok, Y., Anupindi, R., & Akella, R. (1999). Single-period multiproduct inventory models with substitution. *Operations Research*, *47*(4), 632–642.

Ben-Daya, M., & Raouf, A. (1993). On the constrained multi-tem single-period inventory problem. *International Journal of Operations and Production Management*, *13*(11), 104–112.

Cai, L., Chen, J., & Yan, H. (2004). Technical note: single-period two-product inventory model with substitution: solution and properties. *Journal of Systems Science and Systems Engineering*, *13*, 112–123.

Carrillo, J., Vakharia, A., & Wang, R. (2011). Environmental implications for online retailing. *Working Paper*.

Casimir, R. (2002). The value of information in the multi-item newsboy problem. *Omega*, *30*(1), 45–50.

Chen, L., & Chen, Y. (2010). A multiple-item budget-constraint newsboy problem with a reservation policy. *Omega*, *38*(6), 431–439.

Cheng, T.C.E., & Choi, T.M. (2010). Innovative quick response programs in logistics and supply chain management. *International handbooks on information systems*. Berlin: Springer.

Choi, S., Ruszczynski, A., & Zhao, Y. (2011). A multiproduct risk-averse newsvendor with law-invariant coherent measures of risk. *Operations Research*, *59*(2), 346–364.

Chung, C., Flynn, J., & Kirca, O. (2008). A multi-item newsvendor problem with preseason production and capacitated reactive production. *European Journal of Operational Research*, *188*(3), 775–792.

Dutta, P., & Chakraborty, D. (2010). Incorporating one-way substitution policy into the newsboy problem with imprecise customer demand. *European Journal of Operational Research*, *200*(1), 99–110.

Erlebacher, S. (2000). Optimal and heuristic solutions for the multi-item newsvendor problem with a single capacity constraint. *Production and Operations Management*, *9*(3), 303–318.

Gallego, G., & Moon, I. (1993). The distribution free newsboy problem: review and extensions. *The Journal of the Operational Research Society*, *44*(8), 825–834.

Hadley, G., & Whitin, T. (1963). *Analysis of inventory systems*. New jersey: Prentice-Hall.

Khouja, M., & Mehrez, A. (1996). A multi-product constrained newsboy problem with progressive multiple discounts. *Computers and Industrial Engineering*, *30*(1), 95–101.

Khouja, M., Mehrez, A., & Rabinowitz, G. (1996). A two-item newsboy problem with substitutability. *International Journal of Production Economics*, *44*(3), 267–275.

Lau, A., & Lau, H. (1988). Maximizing the probability of achieving a target profit in a two-product newsboy problem. *Decision Sciences*, *19*(2), 392–409.

Lau, H., & Lau, A. (1995). The multi-product multi-constraint newsboy problem: applications, formulation and solution. *Journal of Operations Management*, *13*(2), 153–162.

Lau, H., & Lau, A. (1996). The newsstand problem: a capacitated multiple-product single-period inventory problem. *European Journal of Operational Research*, *94*(1), 29–42.

Lau, H., & Lau, A. (1997). Some results on implementing a multi-item multi-constraint single-period inventory model. *International Journal of Production Economics*, *48*(2), 121–128.

Mahajan, S., & van Ryzin, G. (2001). Stocking retail assortments under dynamic consumer substitution. *Operations Research*, *49*(3), 334–351.

McGillivray, A. R., & Silver, E. A. (1978). Some concepts for inventory control under substitutable demand. *INFOR*, *16*(1), 47–63.

Mieghem, J. V., & Rudi, N. (2002). Newsvendor networks: inventory management and capacity investment with discretionary activities. *Manuacturing Service Operations Management*, *4*(4), 313–335.

Moon, I., & Silver, E. A. (2000). The multi-item newsvendor problem with a budget constraint and fixed ordering costs. *The Journal of the Operational Research Society*, *51*(5), 602–608.

Nagarajan, M., & Rajagopalan, S. (2008). Inventory models for substitutable products: optimal policies and heuristics. *Management Science*, *54*(8), 1453–1466.

Nahmias, S., & Schmidt, C. P. (1984). An efficient heuristic for the multi-item newsboy problem with a single constraint. *Naval Research Logistics Quarterly*, *31*(3), 463–474.

Netessine, S., & Rudi, N. (2003). Centralized and competitive inventory models with demand substitution. *Operations Research*, *51*(2), 329–335.

Niederhoff, J. (2007). Using separable programming to solve the multi-product multiple ex-ante constraint newsvendor problem and extensions. *European Journal of Operational Research*, *176*(2), 941–955.

Ozler, A., Tan, B., & Karaesmen, F. (2009). Multi-product newsvendor problem with value-at-risk considerations. *International Journal of Production Economics*, *117*(2), 244–255.

Parlar, M., & Goyal, S. (1984). Optimal ordering decisions for two substitutable products with stochastic demands. *OpSearch*, 21, 1–15.

Pasternack, B. A., & Drezner, Z. (1991). Optimal inventory policies for substitutable commodities with stochastic demand. *Naval Research Logistics (NRL)*, *38*(2), 221–240.

Qin, Y., Wang, R., Vakharia, A., Chen, Y., & Hanna-Seref, M. (2011). The newsvendor problem: review and directions for future research. *European Journal of Operational Research*, *213*, 361–374.

Rajaram, K., & Tang, C. (2001). The impact of product substitution on retail merchandising. *European Journal of Operational Research*, *135*(3), 582–601.

Shah, J., & Avittathur, B. (2007). The retailer multi-item inventory problem with demand cannibalization and substitution. *International Journal of Production Economics*, *106*(1), 104–114.

Shao, Z., & Ji, X. (2006). Fuzzy multi-product constraint newsboy problem. *Applied Mathematics and Computation*, *180*(1), 7–15.

Shi, J., & Zhang, G. (2010). Multi-product budget-constrained acquisition and pricing with uncertain demand and supplier quantity discounts. *International Journal of Production Economics*, *128*(1), 322–331.

Shi, J., Zhang, G., & Sha, J. (2011). Jointly pricing and ordering for a multi-product multi-constraint newsvendor problem with supplier quantity discounts. *Applied Mathematical Modelling*, *35*(6), 3001–3011.

Silver, E., Pyke, D., & Peterson, R. (1998). *Inventory management and production planning and scheduling*. New York: Wiley.

Smith, S., & Agrawal, N. (2000). Management of multi-item retail inventory systems with demand substitution. *Operations Research*, *48*(1), 50–64.

Vaagen, H., & Wallace, S. W. (2008). Product variety arising from hedging in the fashion supply chains. *International Journal of Production Economics*, *114*(2), 431–455.

Vairaktarakis, G. (2000). Robust multi-item newsboy models with a budget constraint. *International Journal of Production Economics*, *66*(3), 213–226.

Zhang, B., & Du, S. (2010). Multi-product newsboy problem with limited capacity and outsourcing. *European Journal of Operational Research*, *202*(1), 107–113.

Zhang, B., & Hua, Z. (2010). A portfolio approach to multi-product newsboy problem with budget constraint. *Computers & Industrial Engineering*, *58*, 759–765.

Zhang, B., Xu, X., & Hua, Z. (2009). A binary solution method for the multi-product newsboy problem with budget constraint. *International Journal of Production Economics*, *117*(1), 136–141.

Zhang, G. (2010). The multi-product newsboy problem with supplier quantity discounts and a budget constraint. *European Journal of Operational Research*, *206*(2), 350–360.